1. Introduction

Welcome to the series of E-learning module on Forces of Mortality and Expectation of Life Table.

At the end of this session, you will be able to:

- Explain the concept of Forces of Mortality and Expectation of Life Table

Let us start with an introduction:
Death is a principle ‘vital event’ for which the vital statistics are collected and compiled by the vital statistics registration system.
The other principal vital events for which vital statistics are collected and compiled are live births, fetal deaths, marriages and divorces.
Secondly, adoptions, legitimations, annulments and legal separations may be included.

The vital statistics system includes the legal registration, statistical recording and reporting of the occurrence of vital events and the collection, compilation, analysis, presentation and distribution of vital statistics.
The vital statistics employs the registration method of collecting the data on vital events which typically involves the reporting to government officials of events as they occur and the recording of the occurrence and the characteristics of these events.

Broadly speaking death statistics are needed for the purpose of demographic studies and for public health administration.
The most important use of death statistics include:

- Analysis of the present demographic status of the population as well as its potential growth
- Filling the administrative and research needs of the public health agencies in connection with the development, operation and evaluation of the public health programs
- Determination of administrative policy and action in connection with the programs of the government agencies other than those concerned with the public health and filling the need for information on population changes in relation to numerous professional and commercial activities

2. Forces of Mortality

Death statistics are needed to make the analysis of the past population changes which are required for making projections of population and other demographic characteristics.
The forces of mortality at age x is defined as the ratio of instantaneous rate of decrease in 'l' of 'x' to the values of 'lx'.
It is denoted by mu of x and is given by the expression: mu of x is equal to one by 'l' of 'x' into 'd' into 'l' of 'x' by 'd' of 'x' is equal to minus d by d of x into log of 'l' of 'x'.

It gives 'nominal annual rate of mortality', that is the probability of a person of age x exactly dying within the year if the risk of dying is same at every moment of the year as it is during the moment following the attainment of age x.

The relationship between mu of 'x' plus half is equal to 'm' of 'x'.
Let us prove this equation. By definition we have 'L' of 'x' is equal to integral 1 to zero of 'l' of 'x' plus 't' into 'dt'.
Taking the differentiation, we get differentiation of 'L' of 'x' is equal to integral 'd' by 'dx' of 'l' of 'x' plus 't' into 'dt'.
(Assuming the validity of differentiation under the integral sign) is equal to integral 'd' by 'dt' of 'l' of 'x' plus 't' into 'dt', this being possible, since 'l' of 'x' plus 't' is continuous both in 'x' and 't'.
Therefore, 'd' by 'dx' 'L' of 'x' is equal to 'l' of 'x' plus 't' where 't' equal to 0 to 1 is equal to 'l' of 'x' plus 1 minus 'l' of 'x' is equal to minus dx.
It implies that dx by 'L' of 'x' is equal to minus 1 by 'L' of 'x' into d by dx of 'L' of 'x' which implies 'm' of 'x' is equal to 1 by 'l' of 'x' plus half into d by dx of 'l' of 'x' plus half.

Remarks are as follows:

1. This result is obtained under the assumption that deaths are uniformly distributed over the interval x to x plus 1
2. Since 'l' of 'x' is monotonically decreasing function of x, 'd' of 'l' of 'x' by dx is less than equal to zero which implies mu of 'x' is greater than equal to zero. Thus, mu of 'x' is an index of relative rate of growth being 'd' of 'l' of 'x' by dx relative to 'l' of 'x'
3. Approximate expression for mu of 'x'.
   Explicit expression for mu of 'x' can be obtained only if the mathematical form of 'l' of 'x' is known.
   Usually, form of the function 'l' of 'x' is not known and we endeavor to obtain approximate expression for mu of 'x'.
   Assuming that 'l' of 'x' is capable of being expanded as a Taylor's series, we get

'l' of 'x' plus 'h' is equal to 'l' of 'x' plus 'h' into 'l' of 'x' first order differentiation plus 'h' square by 2 factorial 'l' of 'x' second order differentiation plus 'h' cube by 3 factorial 'l' of 'x' third order differentiation and so on.
'L' of 'x' minus 'h' is equal to 'l' of 'x' minus 'h' into 'l' of 'x' first order differentiation plus h square by 2 factorial 'l' of 'x' second order differentiation minus 'h' cube by 3 factorial 'l' of 'x' third order differentiation an so on.
Where 'l' of 'x' to the power 'r' is the r^{th} differential coefficient of 'l' of 'x' with respect to 'x'.
Therefore, 'l' of 'x' plus 'h' minus 'l' of 'x' minus 'h' is equal to 2 into 'h' into 'l' of 'x' first order differentiation plus 'h' cube by 3 factorial 'l' of 'x' third order differentiation plus 'h' to the power

5 by 60 into ‘l’ of ‘x’ to the power v and so on.

Assuming that ‘l’ of ‘x’ third order differentiation and higher order differential coefficient are negligible, on putting ‘h’ is equal to 1, we get, ‘l’ of ‘x’ plus 1 minus ‘l’ of ‘x’ minus 1 is equal to 2 into ‘l’ of ‘x’, then mu of ‘x’ is equal to minus ‘l’ of ‘x’ dash by ‘l’ of ‘x’ is equal to ‘l’ of ‘x’ minus 1 minus ‘l’ of ‘x’ plus 1 by 2 into ‘l’ of ‘x’ is equal to ‘l’ of ‘x’ minus 1 minus ‘l’ of ‘x’ plus ‘l’ of ‘x’ minus ‘l’ of ‘x’ plus 1) by 2 into ‘l’ of ‘x’.
Therefore, mu of ‘x’ is equal to ‘d’ of ‘x’ minus 1 plus dx by 2 into ‘l’ of ‘x’ for x greater than or equal to 1.

A better approximation to mu of ‘x’ is obtained on retaining terms up to the fourth order differential coefficient of ‘l’ of ‘x’ and neglecting higher order differential coefficient.
Thus, on putting ‘h’ equal to 1 and h is equal to 2 respectively in the equation we get correct to ‘l’ of ‘x’ fourth order differentiation.
‘l’ of ‘x’ plus 1 minus ‘l’ of ‘x’ minus 1 is equal to 2 into ‘l’ of ‘x’ first order differentiation plus 1 by 3 into ‘l’ of ‘x’ third order differentiation and ‘l’ of ‘x’ plus 2 minus ‘l’ of ‘x’ minus 2 is equal to 4 into ‘l’ of ‘x’ first order differentiation plus 8 by 3 into ‘l’ of ‘x’ third order differentiation.
Eliminating ‘l’ of ‘x’ between these equations, we get 8 into ‘l’ of ‘x’ plus 1 minus ‘l’ of ‘x’ minus 1 minus ‘l’ of ‘x’ plus 2 minus ‘l’ of ‘x’ minus 2 is equal to 12 into ‘l’ of ‘x’ dash.
Mu of ‘x’ is equal to minus ‘l’ of ‘x’ dash by ‘l’ of ‘x’ is equal to 8 into ‘l’ of ‘x’ plus 1 minus ‘l’ of ‘x’ minus 1 minus ‘l’ of ‘x’ plus 2 minus ‘l’ of ‘x’ minus 2 is equal to 12 into ‘l’ of ‘x’ dash.

4. Estimation of mu of ‘x’ from Mortality table;
Without proof we state the following formula ‘p’ of ‘x’ is equal to probability of survival in the age group ‘x’ to ‘x’ plus 1 is equal to exponential minus integral 0 to 1 mu of ‘x’ plus ‘t’ into ‘dt’
Then integral 0 to 1 mu of ‘x’ plus ‘t’ into ‘dt’ is equal to minus log of ‘p’ of ‘x’ to the base ‘e’.
Then integral on the left hand side represents mean value of mu of ‘x’ in the interval x, x plus 1 .
If assume that the mean value of mu of ‘x’ in x, x plus 1 is mu of ‘x’ plus half, then as an approximation we get, mu of ‘x’ plus half is equal to minus log ‘p’ of ‘x’ to the base.

Alternatively let us consider2 ‘P’ of ‘x’ minus 1 is equal to probability of survival in the age group x minus 1 to x minus 1 plus 2 that is x minus 1, x plus 1 then 2’P’ of ‘x’ minus 1 is equal to exponential of minus integral from minus 1 to 1 mu of ‘x’ plus ‘t’ into dt then, minus integral minus one to one mu of ‘x’ plus ‘t’ dt is equal minus log of 2’P’ of ‘x’ minus 1 to the base ‘e’ is equal to minus log of ‘p’ of ‘x’ minus 1into ‘P’ of ‘x’.
Approximating the integral on the left side by 2 mu of ‘x’, we get mu of ‘x’ is equal to minus half log of ‘p’ of ‘x’ minus 1 into ‘p’ of ‘x’ to the base ‘e’ is equal to minus half log of ‘p’ of ‘x’ plus 1 to the base ‘e’ plus log of ‘p’ of ‘x’ to the base ‘e’.
Hence, we may use any one of the expressions to estimate the value of mu of ‘x’ in the life table.

3. Expectation of Life (Two Concepts in Measuring the Longevity)

Let us now discuss about Expectation of life:

In measuring the longevity two concepts should be distinguished –life span and life expectancy.
Life span tries to establish numerically extreme limit of age in life.
That is the maximum age that human beings as a species could reach under optimum conditions.

There is no known exact figure under this concept.
For purposes of defining the concept more precisely and of excluding rare cases, we might define life span as the age, beyond which, about less than 0.1 percent of the original cohort lives.

We know that very few persons live over a hundred of years; but, owing to lack of precision in records, it is not known exactly whether life span has been increasing, has remained constant or has declined with time.
Life span appears to be about 100 and may not have changed in historical times.

The situation is different with respect to life expectancy, the expected number of years to be lived, on the average.
Sufficiently accurate records have been available for some time for many countries from which estimates have been prepared.
These estimates have generally come from a current life table, although in some instances they have prepared on the basis of the death statistics alone or of census data alone.

According to this concept, longevity has shown a considerable improvement in modern times in most countries.
On the other hand, for many centuries, there was apparently no upward trend in life expectancy and there has been little or no improvement still among some primitive groups.

4. Expectation of Life (Life Table Function)

The expectation of life at birth is the life table function most frequently used as an index of the level of mortality.
It also represents a summarization of the whole series of mortality rates for all ages combined, as weighted by the life table stationary population.

In fact the reciprocal of the expectation of life is equivalent to the crude death rate of the life table population as can be seen from the following derivation.
'M' of 'l' is equal to the total number of deaths by the total population is equal to 'l' naught by 'T' naught is equal to 1 by 'T' naught by 'l' naught is equal to 1 by 'e' naught of zero.

For example the death rate in the life table calculated is 'm' of 'l' is equal to 'l' naught by 'T' naught is equal to one lakh by 69 lakh 89 thousand 30 is equal to 0 point 01431 or 'm' of 'l' is equal to 1 by 'e' naught of zero is equal to 1 by 69 point 89 is equal to 0 point 01431.

The same formulas give the crude death rate of the life table population, and the growth rate of the population is of course zero.
Represented in the formula 'f' of 'l' is equal to 'l' by 't' naught is equal to 'I' by 'e' naught is equal to 'm' of 'l' .
Therefore 'r' of 'l' is equal to 'f' of 'l' minus 'm' of 'l' is equal to zero.

It has been suggested that in some cases, because of the strong effect of the infant mortality rate on the expectation of life at birth, it would be better to use the expectation of life at the age of 1 as a comparative measure of the general level of mortality of a population, perhaps in conjunction with infant mortality rate.

Another life table function frequently used in the expectation of life at age 65.
This value measures mortality at the older ages, the ages where most of the deaths in the developed countries currently occur.

Other life table value used to measure mortality are the probability surviving from birth to age 65 then, 65 of 'p' of zero is equal to 'l' 65 by 'l' naught and the age to which half of the cohort survives, that is the median age at death of the initial cohort assumed in the life table.

The curtate expectation of life, usually denoted by 'e' of 'x' gives the average number of complete years of life lived by the cohort 'l' naught after age x by each of 'l' of 'x' persons attaining that age.
The complete expectation of life usually denoted by 'e' of 'x' of zero measures the average number of years a person of given age 'x' can be expected to live under the prevailing mortality conditions.

It gives the number of years of life entirely completed and includes the fraction of the year survived in the year in which the death occurs, which on the average can be taken to be half

year.
Thus, we have, ‘e’ of ‘x’ of zero is equal to ‘e’ of ‘x’ plus half, since total number of years lived by ‘l’ of ‘x’ persons of age ‘x’ is given by ‘T’ of ‘x’ is equal to integral 0 to infinite of ‘l’ of ‘x plus ‘t’ ‘dt’.

The complete expectation of life of a person attaining age ‘x’ is obtained from the following relation: ‘e’ of ‘x’ of zero is equal to ‘T’ of ‘x’ by ‘l’ of ‘x.
Where, ‘e’ zero of zero is the expectation of life at age 0, is the average age at death or the average longevity of a person belonging to a given community.

5. Expectation of Life (Theorems)

Let us prove the statement that ‘e’ of ‘x’ is equal to summation of ‘l’ of ‘x’ plus n divided by ‘l’ of ‘x’.
In the usual notations, ‘l’ of ‘x’ is the number of persons of age ‘x’ and ‘d’ of ‘x’ is the number of persons who die before attaining the age ‘x’ plus 1, that is ‘d’ of ‘x’ is number of persons dying in the first year without completing one year of life at age ‘x’.
Therefore total number of years lived by ‘d’ of ‘x’ individuals is equal to zero into ‘d’ of ‘x’ is equal to zero.

Similarly ‘d’ of ‘x’ plus 1 is the number of individuals who die between the age period ‘x’ plus 1, ‘x’ plus 2 that is the number of persons who die in the second year after completing 1 year at age ‘x’.
Therefore, the total number of years lived by ‘d’ of ‘x’ plus 1 persons is equal to 1 into ‘d’ of ‘x’ plus 1 is equal to ‘d’ of ‘x’ plus 1.
In general ‘d’ of ‘x’ plus 1 is the number of persons dying ion the age period ‘x’ plus ‘I’, ‘x’ plus ‘I’ plus 1 that is dying in the ‘I’ plus 1 the year after completing ‘I’ years at age ‘x’.

Thus, the total number of years lived by ‘d’ of ‘x’ plus 1 individuals is given by 1 into ‘d’ of ‘x’ plus 1 where ‘I’ is equal to 0, 1, 2, and so on).
Thus, ‘e’ of ‘x’ is equal to average number of years lived by persons of the given age ‘x’ is equal to summation of ‘I’ ‘d’ of ‘x’ plus ‘I’ divided by ‘l’ of ‘x’ is equal to ‘d’ of ‘x’ plus 1 plus 2 into ‘d’ of ‘x’ plus 2 plus 3 into ‘d’ of ‘x’ plus three and so on divided by ‘l’ of ‘x’ which can be written as 1 by ‘l’ of ‘x’ into ‘l’ of ‘x’ plus 1 minus ‘l’ of ‘x’ plus 2 plus 2 into ‘l’ of ‘x’ plus 1 minus ‘l’ of ‘x’ plus 3 plus 3 into ‘l’ of ‘x’ plus 2 minus ‘l’ of ‘x’ plus 4 and so on.
Which in turn is equal to 1 by ‘l’ of ‘x’ into ‘l’ of ‘x’ plus 1plus ‘l’ of ‘x’ plus 2 plus ‘l’ of ‘x’ plus 3 and so on, thus, ‘e’ of ‘x’ is equal to summation of ‘l’ of ‘x’ plus n divided by ‘l’ of ‘x’.

In corollary 1, we have ‘e’ ‘x’ of zero is equal to ‘e’ of ‘x’ plus half is equal to ‘e’ of ‘x’ plus half is equal to ‘l’ of ‘x’ plus 1 plus ‘l’ of ‘x’ plus 2 plus so on divided by ‘l’ of ‘x’ plus half is equal to half ‘l’ of ‘x’ plus ‘l’ of ‘x’ plus 1 plus ‘l’ of ‘x’ plus 2 and so on divided by ‘l’ of ‘x’ which implies that ‘e’ ‘x’ of zero is equal to ‘t’ of ‘x’ by ‘l’ of ‘x’, as a result already given.

In corollary 2, we have that ‘e’ of ‘x’ is equal to summation of ‘l’ of ‘x’ plus n divided by ‘l’ of ‘x’.
It can also be written as ‘l’ of ‘x’ into ‘e’ of ‘x’ is equal to ‘l’ of ‘x’ plus1 plus ‘l’ of ‘x’ plus 2 plus ‘l’ of ‘x’ plus 3 and so on and ‘l’ of ‘x’ plus 1 ‘e’ of ‘x’ plus 1 is equal to ‘l’ of ‘x’ plus 2 plus ‘l’ of ‘x’ plus 3 plus ‘l’ of ‘x’ plus 4.
By subtracting we get, ‘l’ of ‘x’ into ‘e’ of ‘x’ minus ‘l’ of ‘x’ plus 1 into ‘e’ of ‘x’ plus 1 is equal to ‘l’ of ‘x’ plus 1 where ‘l’ of ‘x’ into ‘e’ of ‘x’ is equal to ‘l’ of ‘x’ plus 1into 1 plus ‘e’ of ‘x’ plus 1).
Therefore, ‘l’ of ‘x’ plus l by ‘l’ of ‘x’ is equal to ‘e’ of ‘x’ by 1 plus ‘e’ of ‘x’ plus 1 where, ‘p’ of ‘x’ is equal to ‘e’ of ‘x’ by 1 plus ‘e’ of ‘x’ plus 1.
Also ‘q’ of ‘x’ is equal to 1 minus ‘p’ of ‘x’ is equal to 1 minus ‘e’ of ‘x’ minus ‘e’ of ‘x’ plus 1 divided by 1 plus ‘e’ of ‘x’ plus 1.

Here’s a summary of our learning in this session, where we have understood:

- The concept of Forces of Mortality and Expectation of Life table