1. Introduction

Welcome to the series of E-learning modules on Population Projection using logistic curves.

By the end of this session, you will be able to:

- Explain the Population Projection using logistic curves.

Let us start with an introduction:

A logistic function or logistic curve is a common sigmoid curve, given its name in 1844 or 1845 by Pierre François Verhulst who studied it in relation to population growth.
A generalized logistic curve can model the "S-shaped" behaviour (abbreviated S-curve) of growth of some population P.
The initial stage of growth is approximately exponential; then, as saturation begins, the growth slows and at maturity, growth stops.

Logistic curve is a particular form of complex types of growth curve.
A symmetric logistic curve, also known as Pearl-reed curve is given by, 'y' is equal to time series 'y' of 't' is equal to 'k' divided by 1 plus exponential of 'a' plus 'b' of 't', where 'b' is less than zero, where' a', 'b' and 'k' are constants and 'y' of 't' is the value of the time series at the time 't'.

The above mentioned equation can also be written as 'y' of 't' is equal to 'k' divided by 1 plus 'e' to the power 'a' into 'e' to the power 'bt' is equal to 'k' by 1 plus 'e' to the power 'bt', where 'b' is less than 0.
Also from the equation we have, 1 by 'y' of 't' is equal to 1 by 'k' into 1 plus 'e' to the power of 'a' plus 'bt'.
Is equal to 1 by 'k' plus 1 by 'k' into 'e' to the power of 'a' into 'e' to the power of 'bt' is equal to 'A' plus 'Bc' to the power of 't', where 'A' is equal to 1 by 'k', 'B' is equal to 1 by 'k' into 'e' to the power of 'a', 'c' is equal to 'e' to the power of 'b', are constants.
Thus, the reciprocal of y follows modified exponential law.
Hence, the given time series observations 'y' of 't' will follow logistic law, if their reciprocal 1 by 'y' of 't' follows modified exponential law.

Accordingly, the first differences delta of 1 by 'y' of 't' change by a constant ratio.
In other words, the first difference of the reciprocals of the given observations when plotted on a semi logarithmic graph paper will exhibit a straight line.

2. Derivation and Properties

The derivation of the equation exponential straight line will be, log of ‘y’ of ‘t’ is equal to ‘A’ one plus ‘B’ one into ‘t’ which implies that ‘y’ is equal to ‘y’ of ‘t’ is equal to ‘ab’ to the power ‘t’ is a simple form of the growth curve called the simple exponential.
This form gives differential ‘y’ for time ‘t’ is equal to ‘ab’ to the power ‘t’ log ‘b’ is equal to ‘y’ log ’b’ is equal to alpha ‘y’.
That is the rate of growth of ‘y’ per unit of time is directly proportional to ‘y’.
But in practice this rate of growth cannot be in the same proportion always. It will continue up to certain level, called the level of saturation, after which it starts declining.

Thus, in general we may take differentiation of’ y’ is equal to alpha ‘y’ of beta minus ‘y’; where alpha is greater than zero and beta is greater than zero.
The factor ‘y’ is called the momentum factor which increases with time ‘t’ and the factor beta minus ‘y’ is known as the retarding factor which decreases with time.
When the process of growth approaches the saturation level beta, the rate of growth tends to zero. The principle depicted by the equation is called the Robertson’s law.

We shall now solve as a differential equation in ‘y’ and ‘t’.
We have differential ‘y’ by ‘y’ into beta minus y is equal to alpha differential ‘t’ is equal to one by beta into one by ‘y’ plus one by beta minus ‘y’ into differential of ‘y’ is equal to alpha differential of ‘t’ is equal to one by ‘y’ plus one by beta minus ‘y’ into differential of ‘y’ is equal to alpha beta differential of ‘t’.
Log of ‘y’ by beta minus ‘y’ equal to alpha beta ‘t’ plus gamma, where gamma is the constant of integration.

Therefore, ‘y’ by beta minus ‘y’ is equal to exponential of alpha, beta ‘t’ plus gamma is equal to ‘e’ to the power of alpha beta ‘t’ into ‘e’ to the power of gamma which implies that beta minus ‘y’ is equal to delta ‘y’ ‘e’ to the power of minus epsilon ‘t’ or ‘y’ is equal to beta by 1 plus delta ‘e’ to the power minus epsilon ‘t’, where epsilon is greater than zero. Where, delta is equal to ‘e’ to the power of minus ‘y’ and epsilon is equal to alpha beta greater than zero.
This equation is same as the previous equation even from this equation we will have 1 by ‘y’ is equal to 1 by beta plus delta by beta into ‘e’ to the power of minus epsilon ‘t’ is equal to ‘A’ one plus ‘B’ one into ‘C’ one to the power ‘t’ where ‘A’ one is equal to 1 by beta, ‘B’ one is equal to delta by beta and ‘C’ one is equal to ‘e’ to the power of minus epsilon.

Properties of logistic curve are as follows:
Logistic curve satisfies Robertson’s law,
Given an equation dy by dt equal to alpha into ‘y’ into beta minus ‘y’, we get
‘D’ square ‘y’ by ‘dt’ square equal to alpha into beta minus ‘y’ into ‘dy’ by ‘dt’ minus ‘y’ into ‘dy’ by ‘dt’ which is equal to alpha into beta minus 2y into ‘dy’ by ‘dt’.
The equation can be represented by,
Alpha square into ‘y’ into beta minus ‘y’ into beta minus 2y.

Therefore, ‘d’ square ‘y’ by ‘dt’ square is greater than zero if and only if beta minus 2y is greater than zero which implies y is less than beta by two.
‘d’ square ‘y’ by ‘dt’ square is less than zero if and only if beta minus 2y is less than zero

which implies ‘y’ is greater than beta by two
Similarly, ‘d’ square y by ‘dt’ square is equal to zero at ‘y’ equal to beta by two.

Thus, the logistic curve has an increasing rate for ‘y’ less than beta by 2 and it has a declining rate for ‘y’ greater than beta by 2.
This means that the Logistic curve has a point of inflexion at ‘y’ equal to beta by two.
This point of inflexion is called the Critical point where from the increasing rate of curve starts to decline.
When we draw the graph of the derivative function of ‘y’ versus ‘t’ we get an elongated ‘S’ shape where ‘y’ takes the value zero to beta by 2 to beta.

Note 1:
The logistics curve can be written as follows:
‘Y’ of ‘t’ is equal to ‘L’ divided by one plus exponential of alpha into beta minus ‘t’.
Where, L, alpha and beta are constants.
The curve thus obtained is Concave upwards for the values of ‘t’ less than beta and Convex upwards for values of ‘t’ greater than beta.
The point of inflexion is at the value where ‘t’ is equal to beta and where the ordinate of ‘y’ of ‘t’ is equal to ‘L’ by 2. The curve thus obtained looks like an elongated ‘S’ shaped letter.

Note 2:
From the previous equation,
‘Y’ is equal to ‘y’ of ‘t’ which is equal to ‘k’ divided by one 1 plus ‘e’ to the power of alpha plus ‘bt’ for values ‘b’ less than zero.
Then rate of growth ‘dy’ by ‘dt’ is given as,
‘Dy’ by ‘dt’ is equal to minus ‘k’ divided by one plus ‘e’ to the power of alpha plus ‘bt’ the whole square into ‘b’ into ‘e’ to the power of alpha plus ‘bt’.
Thus, it is equal to minus ‘b’ into ‘k’ by one plus ‘e’ power alpha plus ‘bt’ into one by one plus ‘ power alpha plus ‘bt’ into ‘e’ power alpha plus ‘bt’.
Which is equal to minus ‘b’ into ‘y’ into ‘y’ by ‘k’ into ‘k’ by ‘y’ minus one which is equal to minus ‘b’ into ‘y’ into one minus ‘y’ by ‘k’
Thus, ‘dy’ by ‘dt’ is equal to zero if ‘y’ equal to zero or ‘y’ equal to ‘k’.
K equal to maximum of ‘y’ of ‘t’.

However it is not enough to do the first derivative.
‘D’ square ‘y’ by ‘dt’ square is equal to minus ‘b’ into ‘dy’ by ‘dt’ into one minus ‘y’ by ‘k’ minus ‘y’ by ‘k’ into ‘dy’ by ‘dt’ Which is equal to minus ‘b’ into ‘dy’ by ‘dt’ into one minus 2y by ‘k’.
For Inflexion point, we have ‘d’ square ‘y’ by ‘dt’ square is equal to zero, where ‘y’ is equal to k by 2.
Thus, ‘k’ by 2 is equal to ‘k’ by one plus ‘e’ power alpha plus beta ‘t’ which means,
‘E’ power alpha plus beta ‘t’ is equal to one and alpha plus beta ‘t’ is equal to zero. And ‘t’ is equal to minus ‘a’ by ‘b’.
Therefore ‘y’ of ‘t’ is equal to ‘k’ by one plus ‘e’ power of alpha plus beta into ‘t’ where, ‘b’ is less than zero and k is equal to max of ‘y’ of ‘t’.

3. Method of Three Selected Point

Let us now consider the fitting of the logistic curve.
As already pointed out, the principle of least squares cannot be applied to fit the logistic curve to the given set of data.
We discuss below the various methods of fitting the logistic curve to the given data, that is, The method of three selected point, Yule's method, Hotelling method and method of successive approximation.

Method of three selected point:
The given time series Data is first placed on a graph paper and a trend line is drawn by the freehand method. Three ordinates y1, y2 and y3 are now taken from the trend line corresponding to selected equidistant points of time, say t is equal to t1, t is equal to t2 and t is equal to t3 respectively such that t2 minus t1 is equal to t3 minus t2.

The sum or average or more than one neighboring values can also be taken with advantage. Values must be equidistant. For population data, geometric mean may be used.

Substituting the values of 't' is equal to t1, t2 and t3 in the equation, we get respectively, y1is equal to ,k, by one plus 'e' to the power of 'a' plus 'bt1', y2 is equal to 'k' by one plus 'e' to the power 'a' plus 'bt2', and y3 is equal to 'k' by one plus 'e' to the power 'a' plus 'bt3' .
It also implies, 'a' plus' bt1' is equal to log 'e' of 'k' by y1 minus one; 'a' plus 'bt2' is equal to log 'e' of 'k' by y2 minus one; 'a' plus 'bt3' is equal to log 'e' of 'k' by y3 minus one,
Which implies 'b' into t2 minus t1 is equal to log of 'k' by y2 minus one divided by 'k' by y1 minus one and 'b' of t3 minus t2 is equal to log 'k' by y3 minus one divided by 'k' by y2 minus one.

Since the points are equidistant, that is t2 minus t1 is equal to t3 minus t2, we get log of 'k' by y2 minus 1 divided by 'k' by y1 minus 1 is equal to log 'k' by y3 minus 1 divided by 'k' by y2 minus 1 which implies that 'k' by y3 minus 1 into 'k' by y1 minus 1 is equal to 'k' by y2 minus 1 whole square which implies y2 square of 'k' minus y3 into 'k' minus equal to y1 into y3 of 'k' minus y2 square which implies y2 square of 'k' square minus 'k' into y1 plus y3 plus y1into y3) is equal to y1 into y3 of 'k' square plus y2 square minus 2y2) which implies 'k' square of y2 square minus y1 into y3 is equal to 'k' of y2 square of (y1 plus y3) minus 2 y1 into y2 into y3.

Since k is not equal to zero, k is equal to y2 square into y1 plus y3 minus 2 y1 into y2 into y3 divide by y2 square minus y1 into y3.
Also from the above equation we get respectively, 'b' is equal to 1 by t2 minus t1 log 'e' of k minus y2 into y1 divided by k minus y1 into y2 and alpha is equal to log 'e' of k minus y1 by y1 minus bt1.

Let us take an example to understand fitting of logistic curve.
Given are the three selected points y1, y2 and y3 corresponding to t1=2, t2=30 and t3=58 as follows:
t1=2, y1=55 point 8; t2=30, y2=1 hundred 38 point 6; t3=58, y3=2 hundred 51 point 8.
Fit the logistic curve by the method of selected points. Also obtain the trend values for t=5, 18, 25, 35, 46, 50, 54, 60, 66, 70.

Solution:
Let the equation of the logistic curve be ‘y’ of ‘t’ is equal to ‘k’ by 1 plus ‘e’ to the power of ‘a’ plus ‘bt’
Using the equations we get ‘k’ is equal to y2 square into y1 plus y3 minus 2 y1 into y2 into y3 divided by y2 square minus y one into y three is equal to 39 lakh 87 thousand 987 point 70 minus 23 lakh 48 thousand and 5 point 97 by 19 thousand 209 point 96 minus 8 thousand 470 point 44 is equal to 16 lakh 39 thousand 981 point 72 divide by 10 thousand 739 point 52 is equal to 152 point 7.
‘B’ is equal to log ‘e’ of y1 into k minus y2 divided by y2 into ‘k’ minus y1 into 1 by t2 minus t1 is equal to log ‘e’ 10 of 786 point 78 divided 13 thousand 430 point 34 into loge ‘e’ 10 into 1 by 28 is equal to 2 point 8958 minus 4 point 1280 into 2 point 3026 by 28 is equal to minus 0 point 1013.
Alpha is equal to log “e’ of k by y1 minus 1 minus ‘bt’ is equal to log 1 point 7365 into 2 point 3026 plus 0 point 2026 is equal to 0 point 2396 into 2 point 3026 plus 0 point 2026 is equal to 0 point 7543.
Hence the required equation of the logistic curve is : yt is equal to 152 point 7 by 1 plus ‘e’ to the power of 0 point 7543 minus 0 point 1013t.

Trend values:
Let us take ‘e’ to the power of ‘a’ plus ‘bt’ is equal to mu which implies log mu is equal to ‘a’ plus ‘bt’ which implies mu is equal to 0 point 7543 minus 0 point 1013t, now log mu is equal to log mu by log 10 is equal to log mu by 2 point 3026 is equal to 0 point 7543 minus 0 point 1013t by 2 point 3026.

Figure 1

period (t)	$\log_e \mu = 0.7543 - 0.1013t$	$\log_{10}\mu =$ col2/2.3026	μ = Antilog (3)	$y_t = k/1+\mu$
5	0.2478	0.1076	1.2810	66.944
18	-1.0691	-0.4643	0.3434	113.667
25	-1.7782	-0.7723	0.1690	130.624
35	-2.7912	-1.2122	0.0613	143.880
46	-3.9055	-1.6961	0.0201	149.691
50	-4.3107	-1.8721	0.0134	150.681
54	-4.7195	-2.0496	0.0089	151.353
60	-5.3237	-2.3120	0.0049	151.955
66	-5.9315	-2.5760	0.0027	152.289
70	-6.3367	-2.7520	0.0018	152.426

Finally the trend values ‘yt’ are given by ‘yt’ is equal to ‘k’ by 1 plus ‘e’ to the power of ‘a’ plus ‘bt’ is equal to ‘k’ by 1 plus mu and are obtained in the following table: Column 1 indicates the values of period (t) as 5, 18, 25, 35, 46, 50, 54, 60, 66, and 70.
Column 2 indicates the values of log mu is equal to 0.7543 minus 0.1013’t’ is equal to 0.2478, minus 1.0691, minus 1.7782, minus 2.7912, minus 3.9055, minus 4.3107, minus 4.7195, minus 5.3237, minus 5.9315 and minus 6.33967.
Column three in the table are the values calculated for log mu by log 10 that is the values of column 2 divided by 2.3026.
In the fourth column we get the values of mu by taking the antilog of the values in column 3 and in the last column we have calculated the values of ‘yt’ is equal to ‘k’ divided by 1 plus mu.

4. Yule’s Method and Hotelling Method

Yules method:
Suppose that the value of ‘k’ is approximately known or obtained by other methods, then the logistic curve will contain two parameters ‘a’ and ‘b’, and two variables ‘t’ and ‘yt’.
Hence the principle of least squares can be used to estimate ‘a’ and ‘b’.

We will have, ‘a’ plus ‘bt’ is equal to log of ‘k’ by ‘y’ minus 1 or ‘v’ is equal to ‘a’ plus ‘bt’.
Where ‘v’ is equal to ‘k’ by y minus 1 represents the linear trend between v and t, according to the principle of least squares, the normal equations for estimating a and b are summation ‘v’ is equal to n into a plus ‘b’ summation ‘t’ and summation ‘t’ into ‘v’ is equal to ‘a’ summation ‘t’ plus ‘b’ summation ‘t’ square.

Hotelling method:
A very elegant and ingenious method for fitting a logistic curve is given by Hottelling.
We have differentiation of y is equal to minus ‘by’ into 1 minus ‘y’ by ‘k’ which implies 1 by ‘y’ into ‘dy’ by ‘dt’ is equal to minus ‘b’ into 1 minus ‘y’ by ‘k’.
If the interval is not too large, then as an approximation to 1 by ‘y’, ‘dy’ by ‘dt’, we take 1 by ‘y’, delta ‘yt’ by delta ‘t’.
Thus, we get 1 by ‘yt’, delta ‘yt’ by delta ‘t’ is equal to minus ‘b’ plus ‘b’ by ‘k’ into ‘yt’ or ‘u’ is equal to ‘A’ plus ‘By’ where ‘U’ is equal to 1 by ‘y’ into delta ‘yt’ by delta ‘t’, ‘A’ is equal to minus ‘b’ and ‘B’ is equal to ‘b’ by ‘k’,

Where, ‘A’ and ‘B’ and consequently ‘b’ and ‘k’ can be estimated from the equation by the principle of least squares.
The constant ‘a’ is then obtained by assuming that the curve passes through the mean of Y and the mean of ‘t’.
This is so far the best method, both from the point of simplicity and accuracy. Logistic curve is found very good for measuring the trend of the population data.

5. Method of Successive Approximation

Method of successive Approximation:
If some approximate values of the parameter ‘k’, ‘a’, ‘b’ are known, then a first correction for each of these can be obtained by the principle of least squares. Let us suppose that first correction for ‘k’, ‘a’ and ‘b’ are mu, lambda and delta respectively so that ‘y’ of ‘t’ is equal to k plus mu by 1 plus exponential of ‘a’ plus lambda plus (b plus delta) into ‘t’ is equal to ‘k’ plus mu by 1 plus exponential of ‘a’ plus ‘bt’ into exponential of lambda plus delta ‘t’.

Which is equal to ‘k’ plus mu by 1 plus ‘e’ to the power of ‘a’ plus ‘bt’ into 1 plus lambda plus delta ‘t’, higher posers of lambda and delta being ignored since lambda and delta are very small.
Therefore, ‘y’ of ‘t’ is equal to k plus mu by 1 plus ‘e’ to the power of ‘a’ plus ‘bt’ into 1 plus lambda plus delta ‘t’ into ‘e’ to the power of ‘a’ plus ‘bt’ divided by 1 plus ‘e’ to the power of ‘a’ plus ‘bt’ whole to the power of minus 1 is equal to k plus mu by 1 plus ‘e’ to the power of ‘a’ plus ‘bt’ into 1 minus lambda plus delta ‘t’ into ‘e’ to the power of ‘a’ plus ‘bt’ divided by 1 plus ‘e’ to the power of ‘a’ plus ‘bt’, when higher powers being neglected.

Therefore, ‘y’ of ‘t’ is equal to ‘k’ divided by 1 plus ‘e’ to the power of ‘a’ plus ‘bt’ plus mu divided by 1 plus ‘e’ to the power of ‘a’ plus bt minus k into lambde ‘e’ to the power of ‘a’ plus ‘bt’ by 1 plus ‘e’ to the power of ‘a’ plus ‘bt’ whole square minus ‘k’ into ‘t’ into delta ‘e’ to the power of ‘a’ plus ‘bt’ by 1 plus ‘e’ to the power of ‘a’ plus ‘bt’ whole square ignoring the terms involving lambda delta and mu delta.
The above equation can also be rewritten as say, ‘y’ of ‘t’ is equal to ‘A t’ plus mu ‘Bt’ plus lambda Ct plus delta Dt where At, Bt, Ct, Dt are known as k, a and b. mu lambda and delta can be obtained by the principle of least squares .

If the first approximation is not satisfactory, a second approximation can be obtained by the same method.
Proceeding in this manner we can always arrive at the required accuracy.

Here’s a summary of our learning in this session, where we have understood:

- The Population Projection using logistic curves