1. Introduction

Welcome to the series of E-learning modules on Linear Programming problems-Charne's M-technique (without proof).

At the end of this session, you will be able to:

- Explain the significance of artificial variable technique
- Explain the concept of the Big M method
- Explain the iterative procedure of the Big M-method algorithm
- Explain inserting of slack, surplus and artificial variables
- Explain Two –phase method that solves linear programming problems in two phases

Let us start with an introduction:
In certain situations, discussed below, it is difficult to obtain an initial feasible solution. They arise

(a) When the constraints are of the less than or equal to form then summation 'aij' 'xj' is less than or equal to 'bi' where 'xj' is greater than equal to zero but some right hand side constants are negative (that is 'bi' is less than zero). Then, in this case after adding the non-negative slack variable 'si'. where 'I' is equal to 1, 2, 3 up to m, the initial solution so obtained will be 'si' is equal to minus 'bi' for some 'I', it is not the feasible solution because 'I' violates the non-negativity condition of slack variables (that is 'si' is greater than or equal to zero).

(b) When the constraints are of the greater than equal to form then summation 'aij' 'xj' is greater than or equal to 'bi', 'xj' is greater than equal to zero.
In this case, to convert the inequalities into equation form, adding surplus (negative slack) variables, summation 'aij' 'xj' minus 'si' is equal to 'bi', 'xj' is greater than or equal to zero.
Letting 'xj' is equal to zero, where 'j' is equal to 1, 2, 3 up to 'n', we get an initial solution minus 'si' is equal to 'bi' or 'si' is equal to minus 'bi'.
It is also not a feasible solution as it violates the non-negativity conditions of surplus variables (that is 'si' is greater than or equal to zero).

In this case, we add artificial variables, 'Ai', where 'I' is equal to 1, 2 up to 'm' to get an initial basic feasible solution.
The resulting system of equations: summation 'aij' 'xj' minus 'si' plus 'Ai' is equal to 'bi' 'xj', 'si', and 'Ai' is greater than equal to zero, 'I' is equal to 1, 2 up to 'm' has 'm' equations and (n plus m plus m) variables (that is decision variables, 'm' artificial variables and 'm' surplus variables).
An initial basic feasible solution of the new system can be obtained by equating (n plus 2m minus m) is equal to (n plus m) variables equal to zero.

Thus, the new solution to the given linear programming problem is ‘Ai’ is equal to ‘bi’, where ‘I’ is equal to 1, 2 up to ‘m’, which does not constitute a solution to the original system of equations because the two systems of equations are not equivalent.

Thus, to get back to the original problem, artificial variables must be driven to zero in the optimal solution.
There are two methods for eliminating these variables from the solution. They are the two phase method and the Big-M method or method of penalties.

Following are the Remarks:

1. Artificial variable are only a tool to get the simplex method started. These variables will eventually be equated to zero in the solution in order to attain feasibility in the original problem
2. These variables are added to those constraints with equality and greater than or equal to sign

2. Two-Phase Method (Part-1)

Let us now discuss about Two phase method:

1. In the first phase of this method, the sum of the artificial variables is minimized subject to the given constraints to get a basic feasible solution of the linear programming problem.

2. The second phase minimizes the original objective function starting with the basic feasible solution obtained at the end of the first phase.

Since the solution of the linear programming problem is completed in two phases, this is called the two-phase method.

Advantages of this method:

1. No assumptions on the original system of constraints are made, that is the system may be redundant, inconsistent or not solvable in non-negative numbers.

2. It is easy to obtain an initial basic feasible solution for phase 1

3. The basic feasible solution (if it exists) obtained at the end of phase 1, is used to start phase 2

Steps of the algorithm are as explained below:
Phase 1:
Step 1:

a. If all the constraints in the given linear programming problem is less than or equal type, then phase 2 can be directly used to solve the problem. Otherwise, a sufficient number of artificial variables are added to get a basis matrix (that is, identity matrix)

b. If the given linear programming problem is of minimization, then convert it to the maximization type by the usual method

Step2:
Solve the following linear programming problem by assigning a coefficient of minus 1 to each artificial variables and zero to all the other variables in the objective function and with the basic feasible solution ‘x1’ equal to ‘x2’ is equal to ‘xn’ is equal to zero and ‘Ai’ is equal to ‘bi’ where, ‘I’ is equal to 1, 2 up to ‘m’.

Maximize Z star is equal to summation of minus 1 ‘A’
Subject to the constraints,
Summation of ‘aij’ ‘xj’ plus ‘Ai’ is equal to ‘bi’, where ‘I’ is equal to 1, 2 up to ‘m’ ‘xj’ ,’Ai’ is greater than equal to zero.

Apply the simplex algorithm to solve this linear programming problem. The following three cases may arise at optimality:

i. Maximum Z star is equal to zero and at least one artificial variable is present in the basis with position value. Then no feasible solution exists for the original linear programming problem.

ii. Maximum Z star is equal to zero and no artificial variable is present in the basis. Then the basis consists of only decision variables (x'j s) and hence we may move to phase 2 to obtain an optimal feasible solution to the original linear programming problem.

iii. Maximum Z star is equal to zero and at least one artificial variable is present in the basis at zero value.
Then a feasible solution to the above linear programming problem is also a feasible solution to the original linear programming problem.
Now in order to arrive at the basic feasible solution we may proceed directly to phase 2 or else eliminate the artificial basic variable and then proceed to phase 2.

3. Two-Phase Method (Part-2)

Once an artificial variable has left the basis it has served its purpose and can therefore be removed from the simple table.
An artificial variable is never considered for re-entry into the basis.

Remark:
The linear programming problem defined above is also called auxiliary problem.
The value of the objective function in this problem is bounded from above by zero because the objective function represents the sum of artificial variables with negative unit coefficients.

Thus, the solution to this problem can be obtained in a finite number of steps.
Phase 2
Step 3: Assign actual coefficients to the variables in the objective function and zero to the artificial variables which appear at zero value in the basic at the end of phase I.
That is, the last simplex table of phase I can be used as the initial simples table for phase II.

Then apply the usual simplex algorithm to the modified simplex table to get the optimal solution to the original problem. Artificial variables which do not appear in the basis may be removed.

4. Big-M Method

Following is the explanation on the Big-M method:

The Big-M method is another method of removing artificial variables from the basis.
In this method, we assign coefficients to artificial variables, undesirable from the objective function point of view.
If objective function Z is to be minimized, then a very large positive price called penalty is assigned to each artificial variable.

Similarly, if 'Z' is to be maximized, then a very large negative price also called penalty is assigned to each of these variables.
The penalty will be designated by minus M for a maximization problem and plus M for a minimization problem, where M is greater than 0.

The Big-M Method for solving an linear programming problem can be summarized in the following steps:
Steps of the algorithm are as follows:
Step 1: Express the linear programming problem in the standard form by adding surplus variables and artificial variables.
Assign a zero coefficient to surplus variables and a very large positive number Plus M (minimization case) and minus M (maximization case) to artificial variable in the objective function.

Step 2: The initial basic feasible solution is obtained by assigning zero value to original variables.

Step 3: Calculate the values of 'cj' minus 'zj' in last row of this simplex table and examine these values.

i. If all 'cj' minus 'zj' is greater than equal to zero, then the current basic feasible solution is optimal

ii. If for a column k, 'cj' minus 'zj' is most negative and all entries in this column are negative, then the problem has an unbounded optimal solution

iii. If one or more 'cj' minus 'zj' is less than zero, select the variable to enter into the basic with the largest negative 'cj' minus 'zj' value that is 'cj' minus 'zj' is equal to minimum of c_j minus Z_j, where 'cj' minus 'zj' is less than zero, the column to be entered is called key or pivot column

Step 4:
Determine the key row and key element after identifying the variable to become the basic variable, the variable to be removed from the existing set of basic variable is determined.

For this, each number in ‘xB’ column (that is ‘bi’ values) is divided by the corresponding but positive number in the key column and a row is selected for which this ratio, constant column by key column]is non negative and minimum.
This ratio is called the replacement or exchange ratio. That is ‘xBr’ by ‘arj’ equal to minimum of ‘xBi’ by ‘arj’; where, ‘arj’ is greater than zero.
This ratio limits the number of units of the incoming variable that can be obtained from the exchange.
It may be noted here that division by a negative or by a zero element in key column is not permitted.

Remarks:
At any iteration of the simplex algorithm any one of the following cases may arise:

i. If at least one artificial variable is present in the basis with zero value and the coefficient of ‘M’ in each ‘cj’ minus ‘zj’ (where, ‘j’ is equal to 1, 2 up to ‘n’) value is non-negative, then the given linear programming problem has no solution.
That is, the current basis feasible solution is degenerated

ii. If at least one artificial variable is present in the basis with positive value and the coefficient of ‘M’ in each ‘cj’ minus ‘zj’ (where ‘j’ is equal to 1, 2 up to ‘n’) value is non-negative, then given linear programming problem has no optimum basic feasible solution.
In this case, the given linear programming problem has a pseudo optimum basic feasible solution

5. Example

Let us take an example to understand the solving of a linear programming problem using the penalty method.

Example: Solve the following linear programming problem using the penalty (Big-M) method:
Minimize Z = 5’x’1plus 3’x’2

Subject to the constraints,
2x1plus 4x2 less than equal to 12
2x1 plus 2x2 equal to 10
5x1 plus 2x2 greater than equal to 10
And x1, x2 greater than equal to zero

Solution:
Introduce the slack variable s1, surplus variable s2 and artificial variable A1 and A2 in the constraints of the given linear programming problem. The standard form of the linear programming problem is stated as follows:
Maximize Z is equal to 5x1 plus 3x2 plus zero s1 plus zero s2 plus MA1 plus MA2

Subject to the constraints,
2x1 plus 4x2 plus s1 is equal to 12
2X1 plus 2x2 plus A1 is equal to 10
5X1 plus 2x2 minus s2 plus A2 is equal to 10
And x1, x2, s1, s2, A1, A2 is greater than equal to zero.`

An initial basic feasible solution is obtained by letting x1 equal to x2 equal to s2 equal to zero.
Therefore, the initial basic feasible solution is x1 is equal to 12, A1 is equal to 10, A2 is equal to 10 and Minimum Z is equal to 10M plus 10M is equal to 20M.
Here it may be noted that the columns which correspond to the current basic variable and from the basis (identity) matrix are s1(slack variable), A1 and A2 (both artificial variables).

The initial basic feasible solution is given in table.

Figure 1

		C_j	5	3	0	0	M	M	
C_B	B	b (=X_B)	x_1	x_2	s_1	s_2	A_1	A_2	Min ratio x_B/x_1
0	S_1	12	2	4	1	0	0	0	12/2
M	A_1	10	2	2	0	0	1	0	10/2
M	A_2	10	5	2	0	-1	0	1	10/5
Z=20M		Z_j	7M	4M	0	-M	M	M	
		C_j-Z_j	5-7M	3-4M	0	M	0	0	

'Cj' is equal to 5, 3, 0, 0, M, M. the coefficient of basic variable 'cB' is equal to zero, M, M.
Variables in basis 'B' are s1, A1, A2. The solution values 'b' equal to 'xB' are 12, 10, 10.
The variables x1 are 2, 2, 5.
The variables x2 are 4, 2, 2.
The slack variable s1 are 1, 0, 0.
The surplus variable s2 are 0, 0, minus 1.
The artificial variable A1 are 0, 1, 0 and A2 is 0, 0, 1.
Since the value 'c1' minus 'z1' is equal to 5 minus 7M is the smallest value, therefore, x1 becomes the entering variable.
To decide which basic variable should leave the basis, the minimum ratio is calculated as 12 by 2, 10 by 2 and 10 by 5 which is nothing but 'xB' by 'x1'.
'Zj' is calculated by multiplying the 'x1' variable and the coefficient and taking their summation, so in the 'x1' column we will get 0 into 2 plus M into 2 plus m into 5 is equal to 7M.Similarly we will get the 'Zj' values of the other column as 4M, 0 minus M, M and M.
In the next row we calculate the values of 'cj' minus 'zj' which is equal to 5 minus 7M, 3 minus 4M, 0, M , 0 and 0.

Iteration 1:
Introduce variable 'x1' into the basis and remove 'A2' form the basis by applying the following row operations.
The row solution is shown in the table below:
R3 new is given by R3 old by 5.
R2 new is given by R2 old minus 2R3 new.
R1 new is given by R1 old minus 2R3 new.

Then we get the new table with the following values.

Figure 2

		C_j	5	3	0	0	M	
c_B	B	b (=X_B)	x_1	X_2	s_1	s_2	A_1	Min ratio x_B/x_1
0	S_1	8	0	16/5	1	2/5	0	8/(16/5)
M	A_1	6	0	6/5	0	2/5	1	6/(6/5)
5	x_1	2	1	2/5	0	-1/5	0	2/(2/5)
Z=10+6M		z_j	5	(6M/5)+2	0	(2m/5)-1	M	
		c_j-z_j	0	(-6M/5) +1	0	(-2M/5)+1	0	

The coefficients are 0, m and 5.
The variables in basis are s1, A1 and x1.
The solution values are 8, 6, and 2.
The 'cj' values are the same.
The values of x1 variable are 0, 0 and 1.

The values of x2 variable are 16 by 5, 6 by 5 and 2 by 5.
The values of s1 are 1, 0 and 0.
The s2 values are 2 by 5, 2 by 5 and minus 1 by 5.
The values of A1 are 0, 1, and 0.
The minimum ratios are 8 divided by 16 by 5, 6 divided by 6 by 5 and 2 divided by 2 by 5.
The values of 'z' are 10 plus 6M, the values of 'zj' are 5, 6M by 5 plus 2, 0, 2M by 5 minus 1 and M.
The value of 'cj' minus 'zj' are 0, minus 6M by 5 plus 1, 0, minus 2M by 5 plus 1 and 0.

Iteration 2:
Introduce variable x2 into the basis and remove s1 from the basis by applying the following elementary row operations.
The new solution is shown in the table below:
R1 new is given by R1 old into 5 by 16;
R2 new is given by R2 old minus 6 by 5 R1 new;
R1 new is given by R3 new minus R3 old minus 2 by 5 R1 new.

We will get the table with the following values.

Figure 3

		C_j	5	3	0	0	M	
c_B	B	b (=X_B)	x_1	x_2	s_1	s_2	A_1	Min ratio x_B/x_1
0	x_2	5/2	0	1	5/16	1/8	0	(5/2)/(1/8)
M	A_1	3	0	0	-3/8	1/4	1	3/(1/4)
5	x_1	1	1	0	-1/8	-1/4	0	
Z=5+3M		z_j	5	3	-3M/8 + 5/16	M/4 – 7/8	M	
		c_j-z_j	0	0	3M/8-5/16	-M/4 + 7/8	0	

The coefficients are 0, M and 5.
The variables in basis are x2, A1 and x1.
The solution values are 5 by 2, 3 and 1.
The 'cj' values are the same.
The values of 'x1' variable are 0, 0 and 1.
The values of 'x2' variable are 1, 0 and 0.
The values of 's1' are 5 by 16, minus 3 by 8 and minus 1 by 8.
The 's2' values are 1 by 8, 1 by 4 and minus 1 by 4.
The values of A1 are 0, 1, 0.
The minimum ratio are 5 by 2 divided by 1 by 8), 3 divided by 1 by 4).
The values of 'z' are 5 plus 3M.
The values of 'zj' are 5, 3, minus 3M by 8, M by 4 and M.
The values of 'cj' minus 'zj' are 0, 0, 3M by 8 minus 5 by 16, minus M by 4 plus 7 by 8 and 0.

Iteration 3:
Introduce s2 into the basis and remove A1 from the basis by applying the following new operation
R2 new is given by R2 old into 4;
R1 new is given by R1 old minus 1 by 8R2 new;
R3 new is given by R3 old plus 1 by 4 R2 new.

We will get the table with the following values.

Figure 4

		C_j	5	3	0	0	M
C_B	B	b (=X_B)	X_1	x_2	s_1	s_2	A_1
3	X_2	1	0	1	1/2	0	-1/2
0	S_2	12	0	0	-3/2	1	4
5	X_1	4	1	0	-1/2	0	-1
z = 23		z_j	5	3	-1	0	7/2
		c_j-z_j	0	0	1	0	M-7/2

The coefficients are 3, 0 and 5.
The variables in basis are x2, s2 and x1.
The solution values are 1, 12 and 4.
The 'cj' values are the same.
The values of x1 variable are 0, 0 and 1.
The values of x2 variable are 1, 0 and 0.
The values of s1 are 1 by 2, minus 3 by 2 and minus 1 by 2.
The s2 values are 0, 1 and 0.
The values of A1 are minus 1 by 2, 4 and minus 1.
The value of z is 23.
The values of 'zj' are 5, 3, minus 1, 0 and 7 by 2.
The values of 'cj' minus 'zj' are 0, 0, 1, 0 and M minus 7 by 2.

Figure 5

X_1=4	S_1=0	**Min Z=23**
X_2=1	S_2=12	

From the above table it is observed that all 'cj' minus 'Zj' is greater than equal to zero. Thus, an optimal solution has arrived at with value of variables as x1 is equal to 4, x2 is equal to 1, s1 is equal to 0, s2 is equal to 12 and Minimum Z is equal to 23.

Here's a summary of our learning in this session, where we have understood:

- The Significance of artificial variable technique
- The Concept of the Big M method
- The Iterative procedure of the Big M-method algorithm
- The Inserting of slack, surplus and artificial variables
- The Two –phase method that solves linear programming problems in two phases