1. Introduction

Welcome to the series of E-learning modules on Linear Programming problems by Simplex Algorithm method (without proof).

By the end of this session, you will be able to:
- Solve an linear programing problem by simplex algorithm method
- Understand various important terms such as, basic solution, degenerate etc
- Understand the steps of simplex algorithm for obtaining optimal solution

Let us start with an introduction on Simplex – Algorithm (without proof)

Most real life problems when formulated as on linear programming model have more than two variables and are too large to be interpret with the help of the graphical solution method.
We therefore need a more efficient method to suggest an optimal solution for such problems.

Let us discuss a procedure called simplex method, which is used for solving a linear programming model of such problems.
The method was developed by G B Dantzig in 1947.
The simplex is an important term in mathematics, one that represents an object in an n-dimensional space, connecting n plus one point.

In one dimension, a simplex is a line segment connecting two points, in two dimensions, it is a triangle formed by joining three points; in three dimensions, it is a four sided pyramid, having four corners.
The concept of simplex method is similar to the graphical method. In the graphical method, extreme points of the feasible solution space are examined in order to search for the optimal solution that lies at one of these points.

For linear programming problems with several variables, we may not be able to graph the feasible region but the optimal solution will still lie at an extreme point of the many sided, multi-dimensional figure called n-dimensional polyhedron that represents the feasible solution space.

The simplex method examines the extreme points in a systematic manner repeating the same set of steps of the algorithm until an optimal solution is found. It is for this reason that it is called the iterative method.

Since the number of extreme points (corners or vertices) of the feasible solution space are finite, the method assures an improvement in the value of the objective function as we move from one iteration (extreme point) to another and achieve the optimal solution in a finite number of steps.
The method also indicates when an unbound solution is reached.

2. Standard Form of Linear Programming Problem

Standard form of linear programming problem:
The use of simplex method to solve a linear programming problem requires that the problem can be converted into its standard form.

The standard form of the linear programming problem should have the following characteristics :

1. All the constraints should be expressed as equations by adding the slack or surplus and artificial variables
2. The right hand side of each constraint should be made non-negative if it is not already; this should be done by multiplying both sides of the resulting constraint by minus 1.
3. The objective function should be of the maximization type.

The standard form of the linear programming problem is expressed as:
Optimize (Maximize or Minimize) Z = 'c1''x1' plus 'c2''x2' plus up to Plus 'cn''xn' plus zero s1 plus zero s2 plus up to Plus zero sm.

Subject to the linear constraints,
'a' one one 'x1' plus 'a' one two 'x2' up to plus 'a' one n 'xn' plus s1 is equal to b1, similarly 'a' two one 'x1' plus 'a' two two 'x2' up to plus 'a' two n 'xn' plus s2 is equal to b2 and so on up to 'am1' 'x1' plus 'am2' 'x2' up to plus 'amn' 'xn' plus sm is equal to bm and x1, x2, up to xn, s1, s2, up to sm is greater than equal to zero.

The standard form of the linear programming problem can also expressed in the compact from as follows:
Optimize (Maximize or Minimize) Z = summation 'n', where 'j' is equal to 1 of 'cj' into 'xj' plus summation of 'I' equal to 1 of zero 'si' is the objective function subject to constraints summation 'j' equal to 1 'od' 'n' 'aij' into 'xj' plus 'si' is equal to 'bi' where, 'I' is equal to 1, 2 up to 'm' (conatraints) and 'xj', 'sj' is greater than equal to zero, for all 'I' and 'j' (non – negativity conditions).

In matrix notations the standard form is expressed as: optimize (Maximize or Minimize) Z is equal to 'cx' plus zero 's' subject to the constraints 'Ax' plus 's' is equal to 'b' and 'x', 's' greater than equal to zero, where 'c' is equal to 'c1', 'c2' up to 'cn' is the row vector 'x' is equal to 'x1', 'x2' up to xn to the power T, b is equal to 'b1', 'b2' up to 'bm' to the power 'T' and 's' is equal to 's1', 's2' up to sm are column vectors and 'A' is the 'm' by 'n' matrix of coefficients of variables 'x1', 'x2' up to 'xn' in the constraints.

Remark-1:
The constrained optimization (maximization or minimization) problem, may have

a. No feasible solution, that is there may not exist values ‘xj’ where, j = 1, 2 up to ‘n’ that satisfy every constraints
b. A unique optimal feasible solution
c. More than one optimum feasible solution that is alternative optimum feasible solution
d. A feasible solution for which the objective function is unbounded that is the value of the objective minimization problem by selecting an appropriate feasible solution

Remark-2:
Any minimization linear programming problem can be converted into an equivalent maximization problem by changing the sign of cj’s in the objective function.
That is minimize summation n where j is equal to 1 of ‘cj’ ‘xj’ is equal to maximize summation of ‘nj’ equal to 1 of minus ‘cj’ ‘xj’.

Remark-3:
Any constraints expressed by equality sign may be replaced by two weak inequalities. For example, ‘a’ one one ‘x1’ plus ‘a’ one two ‘x2’ plus so on till ‘a’ one ‘n’ ‘xn’ is equal to b1 is equivalent to following two simultaneous constraints, ‘a’ one one’x1’ plus ‘a’ one two ‘x2’ plus so on ‘a’ one n ‘xn’ is lesser than equal to b1 and ‘a’ one one ‘x1’ plus ‘a’ one two ‘x2’ plus so on ‘a’ 1n xn is greater than or equal to b1.

Remark-4:
Three types of additional variables namely : slack variables (s), surplus variables (-s) and artificial variables (A) are added in the given linear programming problem to convert into the standard form for the following reasons:
a. These variables allow us to convert inequalities into equalities, there by converting the given linear programming problem into a form that is amenable to algebraic solution.

Remark-4:
b. These variables permit us to make a more comprehensive economic interpretation of a final solution
c. Help us to get an initial feasible solution represented by the columns of the identity matrix

The summary of the extra variables to be added in the given linear programming problem in order to convert it into a standard form is given:

First column represents the types of constraints.
Second column represents the extra variables which are needed.
Third & fourth column is the Coefficient of extra variables in the objective function which is maximization of ‘z’ & minimization of ‘z’.
Fourth column represents the presence of extra variables in the initial solution mix.

Then, under less than or equal to condition, a slack variable is added which leads to zero values in both maximum ‘z’ & minimum ‘z’.

Under greater than or equal to condition, a surplus variable is subtracted and an artificial variable is added which leads to value of zero & minus ‘m’ in maximum ‘z’ and zero & plus ‘m’ in minimum ‘z’.
Under equal to condition, Only an artificial variable is added which leads to minus ‘m’ in maximum ‘z’ & plus ‘m’ in minimum ‘z’.
And also there is a presence of extra variables in the initial solution mix under all the three condition.

Remarks:
A slack variable represents an unused resource, either in the form of time on a machine, labour hours, money, warehouse space or any number of such resources in various business problems, since these variables don’t yield any profit, therefore such variables are added to the original objective function with zero coefficients.

A surplus variable represents the amount by which solution values exceed a resource. These variables are also called negative slack variables.
Surplus variables, like slack variables carry a zero coefficient in the objective function.

3. Definitions

Definitions:
Basic solution: Given a system of ‘m’ simultaneous linear equations in ‘n’ (>m) unknowns, ‘Ax’ equal to ‘B’ where, ‘A’ is an ‘m’ by ‘n’ matrix and rank (A) is equal to ‘m’. Let ‘B’ be any ‘m’ by ‘n’ non-singular sub-matrix of ‘A’ obtained by reordering ‘m’ linearly independent columns of ‘A’.
Then, a solution obtained by setting ‘n’ minus ‘m’ variables not associated with the columns of ‘B’ equal to zero, and solving the resulting system is called a basic solution to the given system of equations.

The ‘m’ variables, which may be all different from zero, are called basic variables.
The ‘m’ by ‘m’ non-singular sub-matrix ‘B’ is called a basis matrix and the columns of ‘B’ as basis vectors.
If ‘B’ is the basis sub-matrix, then the basic solution to the system of equations will be ‘XB’ is equal to ‘B’ to the power of minus 1 into ‘b’.

Basic feasible solution: A basic solution to the system Ax is equal to b is called basic feasible if xB is greater than or equal to 0.

Degenerate solution: A basic solution to the system Ax is equal to b is called degenerate if one or more of the basic variables vanish.

Associated cost vector: Let XB be a basic feasible solution to the linear programming problem, maximize Z is equal to cx subject to constraints Ax is equal to b and x is greater than equal to 0.
Then the vector cB is equal to cB1, cB2, and so on to cBm, is called the cost vector associated with the basic feasible solution xB and cB, is coefficient of basic variable xi.

4. Simplex Algorithm (Maximization Case)–Step 1 to 4

Let us now discuss about Simplex Algorithm (maximization case):

The steps of the simplex algorithm for obtaining an optimal solution (if it exists) to a linear programming problem are as follows:

Step 1: Formulation of the mathematical model

1. Formulate the mathematical model of the given linear programming problem
2. If the objective function is of minimization, then convert it into maximization by using the following relationship minimize Z is equal to maximize Z to the power star, where Z to the power star is equal to minus Z
3. Check whether all the bi equal to 1, 2 up to m values are positive.
   If any one of them is negative, then multiply the corresponding constraint by minus 1 in order to make bi greater than 0.
   In doing so, remember to change a less than equal type constraint to a greater than equal type constraint and vice versa
4. Express the mathematical model of the given linear programming problem in the standard form by adding additional variables to the left side of each constraint and assign a zero-cost efficient to these in the objective function
5. Replace each unrestricted variable with the difference of the two non- negative variables; replace each non-positive variable with a new non-negative variable, whose value is the negative of the original variable

Step 2: Set-up the initial solution

Write down the coefficients of all the variables in the linear programming model in a tabular form in order to get an initial basic feasible solution XB is equal to B to the power of minus 1 into b.
The following table is a format of an initial simplex method.

The first row indicates the coefficient of the basic variables ‘c of B’ in the objective function that remains the same in successive simplex tables. These values represent the cost or the profit per unit, to the objective function of each of the variables and are used to determine the variable to be entered into the basis matrix B.
The second row provides the major column headings for the simplex table. Column ‘c of B’, lists the coefficients of the current basic variables in the objective function. These values represent are used to calculate the value of Z when any one unit of any variable is brought into the solution.
Column headed by ‘x of B’ represents the current values of the corresponding variables in the basis.
The identity matrix (or basis matrix) represents the coefficients of slack variables that have been added to the constraints. Each column of the identity matrix also represents a basic variable to be listed in column B.

After having set up the initial simplex table, locate the identity matrix and column variables involved in it.
This matrix contains all zeros except positive elements 1's diagonal.
This identity matrix is always a square matrix and its size is determined by the number of constraints.
The identity matrix so obtained is also called a basis matrix because basic feasible solution is represented by B is equal to 1.
Assign the values of the constants (bi's) to the column variables in the identity matrix because 'x of B' equal to B to the power minus one into b equal to I into b equal to b.

The variables corresponding to the columns of the identity matrix are called basic variables and the remaining ones are non- basic variables.
In general, if a linear programming model has 'n' variables and 'm' which is less than 'n' constraints, then 'm' variables would be basic and 'n' minus 'm' variables are non-basic.
That is simplex algorithm works with basic feasible solution (bfs), which is the algebraic version of extreme points.

A bfs is a feasible solution obtained by choosing one basic variable for each constraint.
The remaining ones are non- basic and have zero value.
However, in certain cases some basic variables may also have zero values. This situation is called degeneracy.

Number 'aij' in the columns under each variable are also called substitution rates or exchange coefficients because these represent the rate at which resource 'I' where, 'I'=1, 2 up to 'm' is consumed by each unit of an activity 'j' where 'j' = 1, 2 up to 'n'.
The values 'zj' represent the amount by which the value of objective function 'Z' would be decreased or increased, if one unit of the given variable is added to the new solution.

Each of the values in the 'cj' minus 'zj' row represents the net amount of increase or decrease in the objective function that would occur when one unit of the variable represented by the column head is introduced into the solution. That is 'cj' minus 'zj' (net effect) is equal to 'cj'(incoming unit profit by cost) minus 'zj' (outgoing total profit by cost) where 'zj' is equal to coefficient of basic variables column into exchange coefficient column 'j'.

Step3: Test for optimality
Calculate the 'cj' minus 'zj' value for all non-basic variables.
To obtain the value of 'zj', multiply each element under the 'variables column (column 'aj' of the coefficient matrix) with the corresponding elements in 'cB' column.

By examine the values of 'cj' minus 'zj', the following three cases may arise:

i. If all 'cj' minus 'zj' less than equal to 0, then the basic feasible solution is optimal
ii. If at least one column of the coefficients matrix (that is 'ak') for which 'ck' minus 'zk', is greater than 0 and all other elements are negative that is. 'ajk' less than 0, then there exists an unbounded solution to the given problem

iii. If at least one ‘cj’ minus ‘zj‘, is greater than 0, and each of these columns have at least one positive element (that is ‘aij’) for some row, then this indicates that an improvement in the value of objective function Z is possible.

Step 4: Select the variable to enter the basis

If case-3 of step 3 holds, then select a variable that has the largest ‘cj’ minus ‘zj‘ value to enter into the new solution.

That is ‘ck’ minus ‘zk‘ is equal to Max of ‘cj’ minus ‘zj‘, where ‘cj’ minus ‘zj‘ is greater than 0, the column to be entered is called the key of pivot column.

Obviously such a variable indicates the largest per unit improvement in the current solution.

Such a variable therefore indicates the largest per unit improvement in the current solution

5. Simplex Algorithm (Maximization Case)–Step 5 to 7

Step 5: Test for feasibility (variable to leave the basis)
After identifying the variable to become the basic variable, the variable to be removed from the existing set of basic variable is determined.
For this, each number in ‘xb’ column (that is ‘bi’ values) is divided by the corresponding but positive number in the key column and a row is selected for which this ratio, constant column by key column, is non-negative and minimum.

This ratio is called the replacement (exchange ratio). That is ‘xbr’ by ‘arj’ equal to minimum of ‘xBi’ by ‘arj’ where ‘arj’ greater than 0.
This ratio limits the number of units of the incoming variable that can be obtained from the exchange.
It may be noted here that division by a negative or by a zero element in key column is not permitted.

The row selected in this manner is called the key or pivot row and it represents the variable which will leave the solution.
The element that lies at the intersection of the key row and key column of the simplex table is called key or pivot element.

Step 6: Finding the new solution:

i. If the key element is 1, then the row remains the same in the new simplex table
ii. If the key element is other than 1, then divide each element in the key row including the elements in ‘xb’ – column 0 by the key element, to find the new values for that row
iii. The new values of the elements in the remaining rows of the new simplex table can be obtained by performing elementary row operations on all rows so that all elements except the key element in the key column are zero

In other words, for each row other than the key row, we use the formula;
Number in new row equal to number in old row plus or minus number above or below key element into corresponding number in the new row, that is row replaced in step 6 & in case 2.
The new entries in ‘cB’ that is, coefficient of basic variables and ‘xB’ that is value of basic variables columns are updated in the new simplex table of the current solution.

Step 7: repeat the procedure:
Go to step 3 and repeat the procedure until all entries in the ‘cj’ minus zj’ row become either negative or zero.
Remark: The flow chart of the simplex algorithm for both the maximization and the minimization linear programming problem is shown here.

Here's a summary of our learning in this session, where we have understood:

- To solve an linear programing problem by simplex algorithm method
- The various important terms such as, basic solution & degenerate etc
- The Steps of simplex algorithm for obtaining optimal solution