1. Introduction & Definitions

Welcome to the series of E-learning modules on Linear Programming problems-solutions by graphical method.

By the end of this session, you will be able to:

- Solve the linear programing problem by graphical method
- Understand various important terms such as,
  - Extreme points
  - Infeasibility
  - Redundancy and
  - Multiple solutions
- Demonstrate them with the help of graphical method
- Interpret the solution of the linear programming model

Let us start with an Introduction:

An optimal as well as feasible solution to an linear programming problem is obtained by choosing among several values of decision variables x1, x2, upto xn, the one set of values that satisfy the given set of constraints simultaneously and also provide the optimal, maximum or minimum value of the given objective function.

For linear programming problems that have only two variables, it is possible that the entire set of feasible solutions can be displayed graphically by plotting linear constraints to locate a best solution.
The technique used to identify the optimal solution is called the graphical solution technique for a linear programming problem with two variables.

In this session, we will discuss two graphical solution techniques or approaches to find the optimum solution to a linear programming problem. That is, extreme point enumeration approach and iso-profit or iso-cost function approach.

Let us get familiarized with the following definitions before working on the linear programming problem solutions.

Solution: Solution values of decision variables 'x' of 'j' where j=1,2 upto n which satisfy the constraints of a general linear programming model is called the solution to that linear programming model.

Feasible solution: Solution values of decision variables 'x' of 'j' where j = 1, 2 upto n which satisfy the constraints and non- negativity conditions of a general linear programming model are said to constitute the feasible solution to that linear programming model.

Basic solution: For a set of 'm' equation in 'n' variables 'n' greater than 'm', a solution obtained by setting n-m variables equal to zero and solving for remaining 'm' equations in 'm' variables is called a basic solution.
The n minus m variables whose value did not appear in this solution are called non basic

variables and the remaining ‘m’ variables are called basic variables.

Basic feasible solution: A feasible solution to a linear programming problem which is also the basic solution is called the basic feasible solution. That is all basic variables assume non-negative values.

Basic feasible solutions are of two types:
(a) Degenerate: A basic feasible solution is called degenerate if at least one basic variable possess zero value
(b) Non-degenerate: A basic feasible solution is called non- degenerate if all m basic variables are non-zero and positive

Optimum basic feasible solution: A basic feasible solution which optimizes, maximize or minimizes, the objective function of the given linear programming model is called an optimum basic feasible solution.

Unbounded solution: A solution which can be increased or decreased by the value of objective function of linear programming problem indefinitely, is called unbound solution.

2. Graphical Solution Methods of LP Problem

Graphical solution methods of linear programming problem are discussed below:
While obtaining the optimal solution to the linear programming by the graphical method , the statement of the following theorems of linear programming is used:

i. The collection of all feasible solution to a linear programming problem constitutes a convex set whose extreme points correspond to the basic feasible solutions
ii. There are finite numbers of basic feasible solutions within the feasible solution space
iii. If the convex set of the feasible solutions of the system Ax is equal to b, x greater than or equal to 0, is a convex polyhedron, then at least one of the extreme points gives an optimal solution
iv. If the optimal solution occurs at more than one extreme point, then the value of the objective function will be the same for all convex combinations of these extreme points

Remarks:

1. A convex set is a polygon and by ‘convex’ we mean that if any two points of the polygon are selected arbitrarily, then a straight line segment joining the two points lies completely within the polygon
2. The extreme points of the convex set give the basic solutions to the linear programming problem

Extreme point enumeration approach:
The solution method for an linear programming problem is divided into five steps:
Step 1: State the given problem in the mathematical form.
Step 2: Graph the constraints, by temporarily ignoring the inequality sign and decide about the area of feasible solutions according to the inequality sign of the constraints. Indicate the area of feasible solutions by a shaded area, which forms a convex polyhedron.
Step 3: Determine the coordinates of the extreme points of the feasible solution space.
Step 4: Evaluate the value of the objective function at each extreme point.
Step 5: Determine the extreme point to obtain the optimum value of the objective function.

Iso-Profit or iso-cost function approach:
The steps of iso-profit or iso-cost function approach are as follows:
Step 1: Identify the feasible region and extreme points of feasible region.
Step 2: Draw an iso-profit or iso – cost line for a particular value of the objective function. The word iso here implies that the iso-profit function is a straight line on which every points has the same total profit.

Step3: Move iso-profit lines parallel in the direction of increasing or decreasing objective function values.
Step 4: The feasible extreme point for which the value of iso-profit is largest or smallest is the optimal solution.

Comparison of two graphical solution methods:

After having plotted the constraints of the given linear programming problem and identified the feasible solution space, select one of the two graphical solution methods and proceed to solve the given linear programming problem.

Comparison between Extreme point method & Iso – profit method has shown below:

Followings are the points under Extreme point method:

- Identify coordinates of each of the extreme or corner points of the feasible region by either drawing perpendiculars on the x-axis and the y-axis or by solving two intersecting equations.
- Compute the profit or cost at each extreme point by substituting that points coordinates into the objective unction.
- Identify the optimal solution at that extreme point with the highest profit in a maximization problem or lowest cost in a minimization problem.

Followings are the points under Iso – profit method:

- Determine the slope x1, x2 of the objective function and then join intercepts to reveal the profit or cost line
- In case of maximization, maintain the same slope through a series of parallel lines, and move the lineup and to the right until it touches the feasible region at only one point. But in case of minimization, move down and to the left until it touches only one point in the feasible region.
- Compute the coordinate of that point touched by the iso-profit or iso-cost line on the feasible region
- Compute the profit or cost

3. Special Cases in Linear Programming

Some special cases in linear programming are discussed below:

Alternative (or multiple) optimal solution:
So far we have seen that the optimal solution of any linear programming problem occurs at an extreme point of the feasible region and the solution is unique, that is one solution yields the same value of the objective function.

However the linear programming problem may have more than one solution yielding the same objective function value.
There are two conditions that should be satisfied so that an alternative optimal solution exists:

i. The given objective function is parallel to a constraint that forms the boundary or edge of the feasible solutions region. In other words, the slope of the objective function is same as that of the constraint forming the boundary of the feasible solutions region and
ii. The constraint should form a boundary on the feasible region in the direction of optimal movement of the objective function. In other words, the constraints should be an active constraint

Remark:
The constraint is said to be active or binding or tight, if at optimally, the left hand side of a constraint equal the right hand side.
In other words, an equally constraint is always active. An inequality constraint may or may not be active.
Geometrically, an active constraint is one that passes through one of the extreme points of the feasible solution space.

An unbounded solution:
When the value of decision variables in linear programming is permitted to increase infinitely without violating the feasibility condition, then the solution is said to be unbounded.
Here the objective function value can also be increased infinitely.
However an unbounded feasible region may yield some definite value of the objective function.

An infeasible solution:
If it is not possible to find a feasible solution that satisfies all the constraints, then the linear programming problem is said to have an infeasible solution or alternatively, inconsistence.
Infeasibility depends solely on the constraints and has nothing to do with the objective function.

4. Example on Maximization Linear Programming

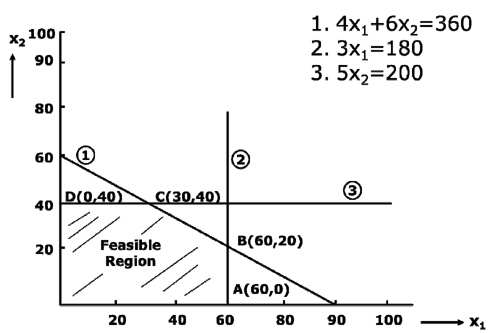
Let us take a few examples to understand the graphical solution method of the linear programming problem:

Maximization linear programming problem:
Use the graphical method to solve the following linear programming problem:
Maximize Z equal to 15’x1’ plus 10’x2’.
Subject to the constraints,

1. 4’x1’ plus 6’x2’ less than equal to 360
2. 3’x1’plus 0’x2’ less than equal to 180
3. 0’x1’ plus 5’x2’ less than equal to 200 and
4. ‘x1’, ‘x2’ ≥ 0

Solution:
Graphical representation is as shown below:

Figure 1



Step 1: State the problem in the mathematical form: in this case the problem is already stated in the mathematical form.
Step2: We shall treat x1 as the horizontal axis and x2 as the vertical axis. Each constraint will be plotted on the graph by treating it as linear equation and it is then that the appropriate inequality conditions will be used to mark the area of feasible solutions.
Consider the constraint 4x1 plus 6x2 less than or equal to 360. Treat this as the equation 4x1 plus 6x2 is equal to 360. The easiest way to plot this line is to find any two points that satisfy the equation and then to draw a straight line through them. The two points are generally the points at which the line intersects the x1 and x2 axis. For example, when x1is equal to 0, we get 6x2 is equal to 360 or x2 is equal to 60.
Similarly when x2is equal to 0, 4x1 is equal to 360, x1 is equal to 90.
These two points are then connected by a straight line as shown in the figure. But the

question is where are these points satisfying 4x1 plus 6x2 is lesser than or equal to 360. Any point above the constraint line violates the inequality condition. But any point below the line does not violate the constraint. Thus, the in-equality and non-negativity condition can only be satisfied by the shaded area that is feasible region.
Similarly, the constraints 3x1is less than equal to 180, and if we take 3x1 is equal to 180 then x1 is equal to 60 in the second equation.
In the third equation 5x2 is less than or equal to 200 and we get x2 is equal to 40 in the third equation and all the three equations are plotted on the graph and are indicated by the shaded area as shown in the figure.
Since all the constraints have been graphed, the area which is bounded by all the constraints lines including all the boundary points is called the feasible region or solution space.

The feasible region is shown by the shaded area OABCD

Figure 2

O =(0,0)	B =(60, 20)	D = (0, 40)
A =(60,0)	C =(30, 40)	

1. Since the optimal value of the objective function occurs at one of the extreme points of the feasible region, it is necessary to determine their coordinates.
   The coordinates of extreme points of the feasible region are: ‘O’ is equal to 0,0; ‘A’ is equal to 60,0; ‘B’ is equal to 60, 20; ‘C’ is equal to 30, 40 and ‘D’ is equal 0, 40.

(31)

2. Evaluate objective function value at each extreme point of the feasible region as shown in the table.

Figure 3

Extreme points	Coordinates (x_1, x_2)	Objective function $Z = 5x_1 + 10x_2$
O	(0,0)	15(0)+10(0)=0
A	(60,0)	15(60)+10(0)=900
B	(60, 20)	15(60)+10(20)=1100
C	(30, 40)	15(30)+10(40)=850
D	(0, 40)	15(0)+10(40)=400

The first column is the extreme points O,A,B,C,D and the second column is the coordinates x1,x2 which are 0,0; 60,0; 60, 20; 30, 40 and 0,40.
The third column shows the objective function value ‘Z’ is equal to 15x1 plus 10x2 so for the first coordinate ‘O’ (0,0), we get the objective function is equal to 0, for second coordinate ‘A’ 60,0, the objective function value will be 900, for third coordinate ‘B’ 60, 20, the objective function value will be one thousand one hundred, for fourth coordinate

‘C’ 30, 40, the objective function value is 850 and the last coordinate ‘D’ 0, 40, the objective function value is 400.

Figure 4

$x_1 = 60$	**Max Z = 1100**
$x_2 = 20$	

3. Since we desire ‘Z’ to be maximum, from the table we can conclude that the maximum value of ‘Z’ equal to one thousand one hundred is achieved at the point extreme ‘B’ 60, 20.
   Hence, the optimal solution to the given linear programming problem is x1 is equal to 60, x2 is equal to 20 and Maximum Z is equal to one thousand one hundred.

Remark: To determine which side of a constraint equation is in the feasible region, examine whether the origin 0, 0 satisfies the constraints.
If it does, then all points on and below the constraints equation towards the origin are feasible points.
If it does not, then all points on and above the constraints equation away from the origin are feasible points.

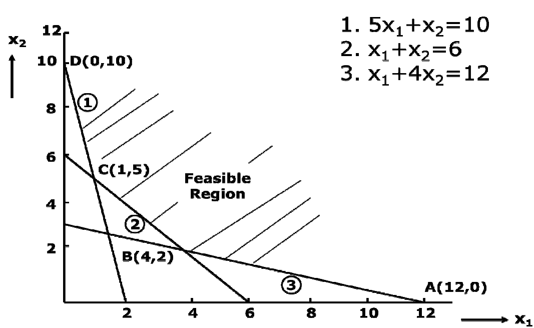
5. Example on Minimization Linear Programming

Minimization linear programming problem:
Use the graphical method to solve the following Linear programming. Minimize Z is equal to 3x1 plus 2x2.
Subject to the constraints,

1. 5x1 plus x2 greater than equal to 10
2. x1 plus x2 is greater than equal to 6
3. x1 plus 4x2 is greater than equal to 12 and
4. x1, x2 is greater than equal to 0

Solution:
Graphical representation is as shown below:

Figure 5



Step 1: State the problem in the mathematical form: in this case the problem is already stated in the mathematical form
Step2: we shall treat x1 as the horizontal axis and x2 as the vertical axis. Each constraint will be plotted on the graph by treating it as linear equation and it is then that the appropriate inequality conditions will be used to mark the area of feasible solutions.
Consider the constraint 5x1 plus x2 greater than or equal to 10. Treat this as the equation 5x1 plus x2 is equal to 10. The easiest way to plot this line is to find any two points that satisfy the equation and then to draw a straight line through them. The two points are generally the points at which the line intersects the x1 and x2 axis.
For example, when x1is equal to 0, we get x2 is equal to 10. Similarly, when x2 is equal to 0, 5x1 is equal to 10, x1 is equal to 2. These two points are then connected by a straight line as shown in the figure.
But the question is where these points are satisfying 5 x1 plus x2 is greater than or equal to 10. Any point above the constraint line violates the inequality condition. But any point below the line does not violate the constraint.

Thus, the in-equality and non-negativity condition can only be satisfied by the shaded area which is feasible region.
Similarly, the constraints x1is greater than equal to 6, and we take x2 is equal to 6 in the second equation.
In the third equation, 4x2 is greater than or equal to 12 and we get x2 is equal to 3 in the third equation and all the three equations are plotted on the graph and are indicated by the shaded area as shown in the figure.
Since all the constraints have been graphed in the area which is bounded by all the constraints lines including all the boundary points is called the feasible region or solution space.

The feasible region is shown by the shaded area ABCD.

Figure 6

A =(12,0)	C =(1, 5)
B =(4, 2)	D = (0, 10)

1. Since the optimal value of the objective function occurs at one of the extreme points of the feasible region, it is necessary to determine their coordinates.
   The coordinates of extreme points of the feasible region are: A is equal to 12, 0; B is equal to 4, 2; C is equal to 1, 5; D is equal 0, 10.
2. Evaluate objective function value at each extreme point of the feasible region as shown in the table.

Figure 7

Extreme points	**Coordinates (x_1, x_2)**	**Objective function $Z= 3x_1+ 2x_2$**
A	(12, 0)	3(12) + 2 (0) = 36
B	(4, 2)	3(4) + 2 (2) = 16
C	(1, 5)	3 (1) + 2 (5) = 13
D	(0, 10)	3 (0) + 2 (10) = 20

The first column is the extreme points A,B,C,D and second column is the coordinates x1,x2 which are 12,0; 4,2; 1,5 and 0, 10.
The third column shows the objective function value Z is equal to 3x1 plus 2x2, so for the first coordinate ‘A’ 12, 0, the objective function value will be 36, for Second coordinate ‘B’ 4, 2, the objective function value will be 16, for third coordinate ‘C’ 1, 5, the objective function value is 13 and for last coordinate ‘D’ 0, 10, the objective

function value is 20.

Figure 8

$x_1 = 1$	Min Z = 13
$x_2 = 5$	

3. Since we desire ‘Z’ to be minimum, from the table we can conclude that the minimum value of ‘Z’ equal to 13 is achieved at the point extreme ‘C’ 1, 5.
   Hence the optimal solution to the given linear programming problem is x1 is equal to 1, x2 is equal to 5 and Minimum ‘Z’ is equal to 13.

Remark:
To determine which side of a constraint equation is in the feasible region, examine whether the origin 0, 0 satisfies the constraints.
If it does, then all points on and below the constraints equation towards the origin are feasible points.
If it does not, then all points on and above the constraints equation away from the origin are feasible points.

Here’s a summary of our learning in this session, where we have understood:

- To Solve an linear programing problem by graphical method
- Various important terms such as extreme points, infeasibility, redundancy and multiple solutions
- To Demonstrate them with the help of graphical method
- To Interpret the solution of the linear programming model