Statistics

Group Replacement of Items with Fail and Staffing Problem

1. Introduction

By the end of this session, you will be able to:

- Make distinctions among various types of failure
- Deriving replacement policy for items that completely fail
- Appreciate the use of replacement analysis in handling problems like “staffing ‘ and ‘equipment’

Let us start with an Introduction on Failure:

The term ‘failure here we discuss in context with the replacement decision. There are two types of failures: gradual failure and sudden failure.

Gradual failure:

It is progressive in nature. That is as the life of an item increases, its operational efficiency also deteriotes.

It results in:

- Increased running (maintenance and operating) costs
- Decrease in its productivity
- Decrease in the resale or salvage value

Mechanical items like pistons, rings, bearings, etc., and automobile tyres fall under this category.

Sudden failure:

This type of failure occurs in items after some period of giving desired service rather than deteriorations while in service. The period of desired service is not constant but follows some frequency distribution which may be progressive, retrogressive or random in nature.

Progressive failure: If the probability of failure of an item increases with the increase in its life, then such a failure is called a progressive failure. For example, light bulbs and tubes fail progressively.

Retrogressive failure: If the probability of failure in the beginning of the life of an item is more, but as time passes the chances of its failure become less, then such failure is said to be retrogressive.

Random failure: In this type of failure, the constant probability of failure is associated with items that fail from random causes such as physical shocks, not related to age. For example, vacuum tubes in air-born equipment have been found to fail at a rate independent of the age of the tube.

Replacement of items that completely fail

It is somehow difficult to predict that particular equipment will fail at a particular time. This uncertainty can be avoided by deriving the probability distribution of failures.
Here it is assumed that the failures occur only at the end of the period, say t. thus, the objective is to find the value of ‘t’ that minimizes the total cost involved for the replacement.

2. Mortality Theorem

Mortality tables are tables used to derive the probability distribution of life span of equipment in question.
Let,

- M of t equal to number of survivors at any time t
- M Of t minus 1 is equal to number of survivors at any time t minus 1
- N is equal to initial number of equipment’s

Then the probability of failure during time period ‘t’ is given by: P of t is equal to M of t minus 1 minus M of t divided by N. The probability that equipment has survived at an age t minus 1 and will fail during the interval t minus 1 to t can be defined as the conditional probability or failure. It is given by: PC of t is equal to M of t minus 1 minus M of t divided by M of t minus 1. The probability of survival to an age t is given by: PS of t is equal to M of t by N.

Mortality theorem:
A large population is subject to a given mortality law for a very long period of time. All deaths are immediately replaced by births and there are no other entries or exits. Show that the age distribution ultimately becomes stable and that the number of deaths per unit time becomes constant and is equal to the size of the total population divided by the mean age at death.

Proof:
Without any loss of generality, it is assumed that death or failure occurs just before the age of k plus 1 years, where k is an integer. That is the life span of an item lies between t is equal to 0 and t is equal to k. let us define, f of t is equal to number of births or replacements at time t, P of x is equal to probability of death or failure just before the age x plus 1, that is failure at time x and summation p of x is equal to 1.

If f of t minus x represents the number of births at time t minus x, t is equal to k, k plus 1, k plus 2, etc, then the age of newly born attain the age x at time t is illustrated in the figure below:

Hence, the expected number of deaths of such newly born before reaching the time t plus 1 that is time t would be: Expected number of deaths is equal to summation p of x into f of t minus x, where t is equal to k, k plus 1, etc. Since all deaths or failures at time t are replaced immediately by births or replacements at time t plus 1, the expected number of births are; f of t plus 1 is equal to summation P of x into f of t minus x, where t is equal to k , k plus 1, etc.

The solution to the difference in the equation in t can be obtained by putting the value f of t is equal to A into alpha to the power of t, where A is some constant then the equation will be A into alpha to the power of t plus 1 is equal to A summation P of x into alpha to the power t minus x.

Dividing both sides of the equation by A into alpha to the power t minus k, we get alpha to the power k plus 1 is equal to summation P of x into alpha to the power of k minus x is equal to alpha to the power k summation P of x into alpha to the power minus x which is equal to alpha to the power k into p of zero plus p of 1 into alpha to the power minus 1 plus p of 2 alpha to the power minus 2 plus or alpha to the power k plus 1 minus p of zero into alpha to the power of k plus p of 1 alpha to the power of k minus 1 plus upto plus p of k is equal to 0. This equation is of degree (k plus 1) and will, therefore, have exactly (k plus 1) roots.

Let us denote the roots in the equation as alpha naught, alpha one, alpha two, up to alpha k. For alpha equal to one, the LHS of the equation becomes one minus P of zero plus p of one plus up to plus p of omega is equal to one minus summation p of x is equal to zero is equal to RHS. Hence one root of the equation is alpha equal to one. Let us denote this root by alpha naught.

This general solution of equation will then be of the form f of t is equal to A naught alpha naught to the power t plus A one alpha one to the power t plus up to plus A k alpha k to the power t which is equal to A naught plus A one alpha one to the power t plus A two alpha two to the power t plus up to plus A k alpha k to the power t, where A naught, A1, A2, up to Ak are constant whose values are to be calculated.

Since one of the roots of the equation are alpha naught equal to 1 is positive, according to the Descrate's sign rule all other roots alpha one, alpha two up to alpha k will be negative and their absolute value would be less than unity, that is absolute alpha i less than 1 , where I is equal to 1, 2, up to k. It follows that the value of these roots tends to zero as t tends to infinite, with the result that it becomes f of t is equal to A naught.

This indicates that the number of deaths as well as births becomes constant at any time. Now the problem is to determine the value of the constant A naught. For this we can proceed as follows. Let us define: g of x is equal to probability of survival for more than x years, g of x is equal to 1 minus probability (survivor will die before attaining the age x) is equal to 1 minus p of zero plus p of one plus up to plus p of x minus 1.

Obviously, it can be assumed that g of zero is equal to 1. Since the number of births as well as deaths has become constant and equal to A naught, the expected number of survivors of age x is given by A naught into g of x. As deaths are immediately replaced by births, size N of the population remains constant.

That is, n is equal to A naught summation g of x or A naught is equal to N by summation g of x. the denominator in the equation represents the average age at death, this also can be proved as follows; from finite differences, we know that, Delta x is equal to x plus 1 minus x is equal to 1, summation f of x into delta h of x is equal to f of b plus 1 into h of b plus 1 minus f of a into h of a minus summation h of x plus 1 into delta f of x.

Therefore we can write, summation g of x is equal to summation g of x into delta x is equal to g of x into x minus summation x plus 1 into delta g of x is equal to g of k plus 1 into k plus 1 minus g of zero into zero minus summation x plus 1 into delta g of x, but g of k plus 1 is equal to 1 minus p of zero plus P of 1 plus p of 2 plus up to plus P of k is equal to zero.

Since no one can survive for more than k plus 1 years of age and delta g of x equal to g of x plus 1 minus g of x is equal to 1 minus p of 0 minus p of 1 minus up to minus p of x minus 1 minus p of 0 minus p of 1 minus up to minus p of x minus 1 equal to minus p of x.
Substituting the value of g of k plus 1 and delta into g of x in the equation we get summation g of x is equal to summation x plus 1 p of x is equal to mean age at death.
Hence we get A naught is equal to N by average age at death.

3. Replacement Policy

Individual replacement policy: Under this policy an item or equipment is replaced just after its failure in the given system. This ensures smooth running of the system.

Group replacement policy: Sometimes just after the complete breakdown of a system, the immediate replacement of the items may not be available. This may result in heavy losses. In such circumstances a group replacement policy can be adopted.

Under this policy items are replaced:

- Individually as and when they fail during a specified time period
- In groups at the end of some suitable time period, without waiting for their failure, with the provision that if any item fails before the time specified, it may be replaced individually

Here, we have to see two things namely: Rate of individual replacement during the specified time period and Total cost incurred for individual as well as group replacement during the specified time.

Obviously, the decision- maker will call a time period optimal for which the total cost incurred is the minimum. In order to calculate this optimal time period for replacement he has to keep record of, probability of failure, loss incurred due to these failures, cost of individual replacement and cost of group replacement.

Remark: The group replacement policy is suitable for a large number of identical low cost of items that are likely to fail with age and for which it is difficult as well as not justified to keep the record of their individual ages.

4. Group Replacement Policy - Theorem & Example

The rate of replacement and total cost involved in the group replacement is based on the following theorem: Theorem: (group replacement policy)

a. Group replacement should be made at the end of the period, t, if the cost of the individual replacement for the period t is greater than the average cost per period through the end of period t

b. Group replacement is not advisable at the end of the period t if the cost of individual replacements at the end of period t minus 1 is less than the average cost per period through the end of period t

Proof: Let us consider the following notations:
n is the total number of items in the system.
F of t is number of items failing during time t.
C of t is the total cost of group replacement until end of period t.
C1 is unit cost of replacement in a group.
C2 is unit cost of individual replacement after time t, that is failure.
L is maximum life of any item.
P of t is the probability of failure of any item at age t.

Rate of replacement at time t : the number of failures at any time t is, F of t is equal to np of t plus summation p of x into F of t minus x, where t is less than equal to L, summation p of x into F of t minus x, where t is greater than L.

Cost of replacement at time t: the cost of group replacement after time period t is given by: C of t is equal to nC1 plus C2 summation F of x where nC1 is the cost of replacing the items as s group and C2 summation F of x is the cost of replacing the individual failures at the end of t minus 1 periods before the group is replaced again.

The average cost per unit period is then given by: C of t by t is equal to nC1 by t plus C2by t summation F of x. For optimal replacement period t, the value of average cost per unit period, given by the above equation, should be the minimum. The condition for minimum C of t by t is: delta C of t by t minus 1 less than 0 less than delta C of t by t. Now for delta C of t by t greater than 0, we have delta C of t by t is equal to C of t plus 1 by t plus 1 minus C of t by t greater than 0.

From the above equation we get: C of t plus 1 by t plus 1 minus C of t by t is equal to nC_1 into 1 by t plus 1 minus 1 by t plus C2 summation F of x into 1 by t plus 1 minus 1 by t plus C_2 F of t by t plus 1 is equal to minus nC1 minus C2 summation F of x plus t C2 F of t by t of t plus 1. For C of t plus 1 by t plus 1 minus C of t by t greater than 0, it is necessary that: t C2 F of t greater than nC1 plus C2 summation F of x into C2 F of t greater than nC1 plus C2 summation F of x by t.

Similarly for delta C of t by t is equal to C of t minus 1 by t minus 1 minus C of t by t greater than 0, the following condition can be derived C2 into F of t minus 1 less than nC1 plus C2 summation F of x by t minus 1. The inequalities in both the above equations describe the necessary condition for optimal replacement.

So the expression nC1 plus C2 summation F of x by t represents the average cost per period if all items are replaced at the end of period t, whereas, expression C2 plus F of t represents the cost for the r^{th} period if group replacement is not made at the end of the period t.

5. Staffing Problem

Other replacement problem: A few replacement situations that are different from the situations discussed earlier are staffing problem.Staffing problem are replacement problems that are related to human beings working in an organization.

The principles of replacement may be applied to formulate some useful recruitment and promotion policies for the staff working in an organization. For this we assume that the life distribution for the service of staff in the organization is already known.

Let us take an example to understand the replacement problem in the staffing of an organization:
An airline requires 200 assistant hostesses, 300 hostesses and 50 supervisors. Women are recruited at the age of 21, and if still in service retire at 60. Given the following life table, determine (a) how many women should be recruited each year (b) at what age should the women be promoted?

First, third, fifth, seventh & ninth row represents the age from 21 to 59. Second, forth, sixth, eight & tenth row represents the number in service of respective ages.

Solution: If one thousand women had been recruited each year for the past 39 years, then the total number of recruited at the age of 21 and those serving up to the age of 59 is six thousand four eighty. Total number of women recruited in the airline are 200 plus 300 plus 50 is equal to 550.

Number of women to be recruited every year in order to maintain strength of 550 hostesses is equal to 550 into one thousand by six thousand four eighty is equal to 85 approximately.

If the assistant hostesses are promoted at the age of x, then up to age (x - 1), 200 assistant hostesses will be required. Among 550 women, 200 are assistant hostesses. Therefore, out of a strength of thousand there will be: 200 into thousand by five fifty is equal to three sixty four assistant hostesses. But from the life table given in the question, this number is available up to the age of 24 years. Thus, the promotion of assistant hostesses is due in the 25^{th} year.

Since out of the 550 recruitments only 300 hostesses are needed, if one thousand girls are recruited, then only one thousand into three hundred by five hundred is equal to five forty five approximately, will be hostesses. Hence, the total number of hostesses and assistant hostesses in a recruitment of 100 will be five forty five plus three sixty four is equal to nine naught nine.

This means only, one thousand minus nine hundred nine is equal to 91 supervisors are required. But from the life table this number is available up to the age of 46 years. Thus the promotion of hostesses to supervisors will be in the 47^{th} year.

Here's a summary of our learning in this session, where we have understood:

- To Make distinctions among various types of failure
- To Derive the replacement policy for items that completely fail

To Appreciate the use of replacement analysis in handling problems like "staffing" and "equipment"