# 1. Introduction & Crude Death Rate

Welcome to the series of E-learning modules on Linear Programming Problems-Formulations

At the end of this session, you will be able to know:

- The Concept of Linear Programming Problems-Formulations
- The Structure and assumptions of Linear Programming problems-Formulations
- The General mathematical model of LP problem
- The LP problem model formulation

Let us start with an Introduction:
Linear programming is the general technique of optimum allocation of ‘scarce’ or ‘limited’ resources, such as labour, material, machine ,capital, energy, etc. to several competing activities, such as products, services, jobs, new equipment’s, projects etc. on the basis of a given criterion of optimality.

The term ‘limited’ here is used to describe availability of scarce resources during planning period and the criterion of optimality generally is either performance, return on investment, utility, time distance, etc.

George B Dantzig, while working with the US Air Force during World War II, developed this technique, primarily for solving military logistics problems.
But, now, it is being used for solving a wide range of problems from diet planning, oil refining, education, energy planning, pollution control, transportation planning and scheduling to research & development and almost in all functional areas of management  such as production , finance, marketing and personnel.

Let us now understand the two words, linear and programming.
The word linear is used to describe the proportionate relationship of two or more variables in a model.
Thus, a given change in one variable will always cause a resulting proportional change in another variable.
For example: Doubling the investment on a certain project will exactly double the rate of return.

The word programming here is used to specify a sort of planning that involves the economic allocation of limited resources by adopting a particular course of action or strategy amongst various alternative strategies to achieve the desired objective.

Out of several courses of action available, the best or optimal is selected.
A course of action is said to be most desirable or optimal if it optimizes some measure of criterion of optimally such as profit, cost, rate of return, time, distance, utility, etc.

# 2. Structure of Linear Programming Problem

Let us now discuss about Structure and assumptions of an linear programming problem:

In order to formulate a linear programming problem, identifying and interpreting the optimal solutions, the general characteristics and assumptions of the linear programming problem should be known.

The Structure of Linear programming is as follows:
The general structure of any linear programming model consists essentially of three components.

1. The activities and their relationships: The activity values represent the extent to which each activity is performed. These are represented by X1, X2, X3 up to Xn.

For example, in a product- mix problem the activities of interest are the production of several products under considerations. These activities are also known as decision variables because they are under the decision maker's control.

These decision variables, usually interrelated in terms of consumption of limited resources, require simultaneous solutions. All decisions variables are continuous, controllable and non-negative.
That is, X1 greater than equal to zero, X2 greater than equal to zero upto Xn greater than equal to zero.

2. The objective function: The objective function of each Linear programming problem is a mathematical representation of the objective in terms of a measurable quantity such as profit, cost, revenue, distance, etc. It is represented in one of two forms:

Optimize (Maximize or Minimize) Z equal to c1 into X1 plus c2 into X2 plus upto 'cn' into 'Xn'.
Where, 'Z' is the measure of performance variable or optimum is a function of X1, X2, X3, upto Xn to the measure of performance of 'Z'.
The optimal value of a given objective function is obtained by the graphical method or simplex method.

3. The constraints: There are always certain limitations on the use of limited resources, for example, labour, machine, raw material, space, money, etc.
   Such constraints must be expressed as linear equalities or inequalities in terms of decision variables. The solution of a Linear Programming model must satisfy these constraints.

# 3. Basic Assumptions of Linear Programming

Basic assumptions of Linear Programming are as follows:

Certainty: In all Linear programming models, it is assumed that all model parameters such as availability of resources, profit contribution of a unit of decision variable and resource consumption by a unit of decision variable must be known and constant.

In some cases, there may be either random variables represented by a known distribution which can be either general distribution or statistical distribution methods.
Using the given parameters the problem can be solved by a stochaistic Linear Programming model or parametric programming.

Divisibility: The solution values of decision variables and resources are assumed to have either whole number, which are integers or mixed numbers which can be either integer or fractional.

If only integer variables are desired like number of employees, types or number of machines used, then the integer programming method may be applied to get the desired values.

Additivity: The value of the objective function for the given values of decision variables and the total sum of resources used, must be equal to the sum of the contributions, be it profit or cost, earned from each decision variable and the sum of the resources used by each decision variable respectively.

For example, if the total profit earned by sale of two products A and B must be equal to the sum of the profits earned separately from A and B.
In the same way, the amount of a resources consumed by A and B must be equal to the sum of the resources used for A and B individually.

Linearity: All relationships in the Linear Programming model are linear. This is applicable in both objective function and in case of constraints.
Ideally in Linear programming given a decision variable, the amount of particular resource and its contribution to the cost in objective function must be directly proportional to its amount.

If the decision variable ‘x’ is equal to 5, then the resource consumed is 5 into ‘a’ of ‘ij’ and if the value of ‘x’ is equal to 10, then the consumption would be 10 into ‘a’ of ‘ij’.
Here ‘a’ of ‘ij’ represents the amount of resource ‘I’ used for an activity ‘j’ where ‘j’ becomes the decision variable.

# 4. General Mathematical Model & Guidelines on Linear Programming

General Mathematical model of Linear Programming problem:

The general linear programming problem with ‘n’ decision variable and ‘m’ constraints can be stated in the following form:
Find the values of decision variables x1, x2, x3, upto xn so as to
Optimize (minimize or maximize) Z equal to c1 into x1 plus c2 into x2 plus c3 into x3 plus upto ‘cn’ into ‘xn’.

We can express the linear constraints as
‘A’ one one into x1 plus ‘a’ one two into x2 plus upto ‘a1n’ into ‘xn’ (less than equal to, equal to and greater than equal to) b1
‘A’ two one into x1 plus ‘a’ two two into x2 plus upto ‘a2n’ into ‘xn’ (less than equal to, equal to and greater than equal to) b2

‘am1’ into x1 plus ‘am2’ into x2 upto ‘amn’ into ‘xn’(less than equal to, equal to and greater than equal to) ‘bm’
and x1, x2 upto Xn is greater than equal to Zero

Summarizing, we can express the resulting formula as,
Optimize (Max or Min) Z equal to Summation ‘j’ ranges from 1 to n of ‘cj’ into ‘xj’ - (that is, Objective Function)

Subject to the linear constraints:
Summation ‘j’ ranges from 1 to n of ‘a’ of ‘ij’ into ‘xj’ (less than equal to, equal to and greater than equal to)’bi’.
Where, ‘I’ is equal 1,2,3 upto ‘m’ (Constraints)
And ‘xj’ greater than equal to zero, where ‘j’ is equal to 1,2,3 upto ‘n’ (satisfying the non-negativity conditions)

In this equation ‘cj’ are the coefficients representing the per unit contribution of decision variable ‘xj’, to the value of the objective function.
The ‘a’ of ‘ij’ are called the Technological coefficient or input output coefficient.
These represent the amount of resources consumed per unit of variable ‘a’ of ‘ij’.
In the given constraints, ‘a’ of ‘ij’ can be positive, negative or zero. The ‘a’ of ‘ij’ represents the Total availability of the $i^{th}$ resource.

The term resource is used in a very general sense to include any numerical value associated with the right hand side of a constraint.
It is assumed that ‘bi’ is greater than equal to zero for all values of ‘i’.
However if any ‘bi’ is less than zero, then both sides of constraint ‘i’ can be multiplied by minus one to make ‘bi’ greater than zero and reverse the inequality of the constraint.

In the general LP problem, the expression (less than equal to, equal to and greater than equal to)means that in any specific problem each constraint may take only one of the three possible forms:

i. Less than or equal to
ii. Equal to
iii. Greater than or equal to

Guidelines on Linear Programming model formulation:

Different steps are involved in formulation of Linear Programming model
Step 1: Define Decision Variables

i. Express each constraint in words and check whether the constraint is of the form (greater than equal to - at least as large as), (less than equal to- no longer than) or (equal to- exactly equal to)
ii. Express the objective function in words
iii. Verbally identify the decision variables

If there are several decision alternatives available, then to identify the decision variables you have to ask yourself the question – What decision must be made in order to optimize the objective function.
Once the above 3 steps are done, decide the symbolic notation for the decision variables and specify their units of measurement.
Such specification of units of measurement would help in interpreting the final solution of the LP problem.

Step 2: Formulate the Constraints
Formulate all the constraints imposed by the resource availability and express them as linear equality or inequality in terms of the decision variables defined in step 1.
These constraints define the range within which values of decision variables can lie.
Wrong formulation can lead to either solutions which are not feasible or excluding some solution which are really feasible and possibly optimal.

Step 3: Formulate the Objective Function
Define the objective function and determine whether the objective function is to be maximized or minimized.
Then, express it as a linear function of decision variables multiplied by their profit or cost contributions.
The following are certain examples of LP model formulation that you can use to.

# 5. Examples

Let us take an example-1 to understand the linear programming problem.
A company engaged in producing tinned food, has 300 trained employees on the rolls each of whom can produce one ‘can’ of food in a week.
Due to the developing taste of public for this kind of food, the company plans to add the existing labour force by employing 150 people, in a phased manner, over the next five weeks.
The new comers would have to undergo a two week training programme before being put to work.

The training is to be given by employees from among the existing ones and it is known that one employee can train three trainees.
Assume that there would be no production from the trainers and the trainees during training period as the training is off the job.
However, the trainees would be remunerated at the rate of Rupees 300 per week, the same rate as for the trainers.

The company has booked the following number of cans to supply during the next five weeks:

**Figure 1**

| Weeks | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| No. of cans | 280 | 298 | 305 | 360 | 400 |

In the table, first row represents the week and the second row represents the number of cans.

Assume that the production in any week would not be more than the number of cans ordered, so that every delivery of the food would be ‘fresh’.
Formulate a LP model to develop a training schedule that minimizes the labour cost over the five periods.

Solution:
LP model formulation; the data of the problem is summarized as given below:

i) Cans supplied are as shown below

**Figure 2**

| Weeks | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| No. of cans | 280 | 298 | 305 | 360 | 400 |

As given in the table, in the first week 280 cans will be supplied, in the second week, 298 cans will be supplied, in the third week 305 cans and in the fourth & fifth week, 360 & 400 cans will be supplied respectively.

ii) Each trainee has to undergo a two-week training
iii) One employee is requires to train three trainees
iv) Every trained worker producing one can per week, but no production from trainers and trainees during training
v) Number of employees to be employed is equal to 150
vi) The production in any week not to exceed the cans required
vii) Number of weeks for which newcomers would be employed are 5, 4, 3, 2 and 1.

From the given information you may observe the following facts:

a) Workers employed at the beginning of the first week would get salary for all the five weeks, those employed at the beginning of the second week would get salary for four weeks, those employed for the third week will get salary for three weeks, those employed for two weeks will get salary for two weeks and the one employed for the last week will get the salary for one week
b) The value of the objective function would be obtained by multiplying it by 300 because each person would get a salary per week
c) Inequalities have been used in the constraints because some workers might remain idle in some week(s).

Decision variables:
Let x1, x2, x3, x4, x5 equal to number of trainees appointed in the beginning of the week 1, 2, 3, 4 and 5 respectively.
The LP model is given by:
Minimize (total labour force) Z=5'x1' plus 4'x2' plus 3'x3' plus 2'x4' plus 'x5' subject to constraints.

i) Capacity constraints :
300 minus 'x1' by 3 is greater than or equal to 280, next 300 minus x1 by 3 minus x2 by 3 is greater than or equal to 298, next 300 plus x1 minus x2 by 3 minus x3 by 3 is greater than or equal to 305, next 300 plus x1 plus x2 minus x3 by 3 minus x4 by 3 is greater than or equal to 360 and 300 plus x1 plus x2 plus x3 minus x4 by 3 minus x5 by 3 is greater than or equal to 400

ii) New recruitment constraints:
x1 plus x2 plus x3 plus x4 plus x5 is equal to 150 and x1, x2, x3, x4, x5 is greater than equal to 0.

Example -2 :
A company has two grades of inspectors 1 and 2, who are assigned for a quality control inspection.
It is required that, at least 2000 pieces be inspected per 8 hour day.
A grade - 1 inspector can check pieces at the rate of 40 per hour, with an accuracy of 97 percent.
A grade-2 inspector checks at the rate of 30 pieces per hour with an accuracy of 95 percent.

**Figure 3**

| Inspector | Wage rate/ hour | Error cost | No. of inspectors |
|---|---|---|---|
| Grade-1 | Rs.5 | Rs.3 | 9 |
| Grade-2 | Rs.4 | Rs.3 | 11 |

The wage rate of a grade 1 inspector is Rupees 5 per hour while that of a grade 2 inspector is Rupees 4 per hour.
An error made by an inspector costs Rupees 3 to the company.
There are only nine grade-1 inspectors and eleven grade-2 inspectors available in the company.
The company wishes to assign work to the available inspectors so as to minimize the total cost of the inspection.
Formulate this problem as a linear programming model.

Solution:

**Figure 4**

| | Grade-1 | Grade-2 |
|---|---|---|
| No. of inspectors | 9 | 11 |
| Rate of checking | 40 piece/hour | 30 piece/hour |
| Accuracy in checking | 1-0.97 = 0.03 | 1-0.95 = 0.05 |
| Cost of accuracy in checking | Rs.3/piece | Rs.3/piece |
| Wage rate per hour | Rs.5 | Rs.5 |
| Duration of inspection | 8 hours/day | |
| Total pieces which must be inspected | 2000 | |

Linear Programming model formulation; the data of the problem is summarized as follows:-

i) Number of inspectors are 9 in grade-1 and 11 in grade-2

ii) Rate of checking is 40 piece per hour for grade-1 inspector and 30 pieces per hour for grade-2

iii) Accuracy in checking is 1 minus 0 point 97 equal to 0 point 03 for grade-1 and 1 minus 0 point 95 is equal to 0 point 05 for grade-2

iv) Cost of accuracy in checking is Rupees 3 per piece for grade-1 inspectors and grade-2 inspectors

v) Wage rate per hour is Rupees 5 for both the grade-1 and grade-2 inspector

vi) Duration of inspection is equal to 8 hours per day and

vii) Total pieces which must be inspected is equal to two thousand

Decision variables: Let x1, x2 is equal to number of grade 1 and 2 inspectors to be assigned for inspection, respectively.

The linear programming model:
Hourly cost of each grade 1 and 2 inspectors can be computed as follows:

For inspector grade 1:  5 plus 3 into 40 into zero point 03 is equal to Rupees 8 point 60.
For inspector grade 2: 4 plus 3 into 30 into zero point 05 is equal to Rupees 8 point 50.

Based on the given data the linear programming problem can be formulated as follows: minimize ( daily inspection cost) Z is equal to 8 whole into 8 point 60'x1' plus 8 point 50'x2' is equal to 68 point 80'x1' plus 68'x2'.

subject to constraints:

i) Total number of pieces that must be inspected in an 8 hour per day constraints. That is, 40 into 8'x1' plus 30 into 8'x2' greater than or equal to two thousand
ii) Number of inspectors of grade 1 and 2 available constraint 'x1' is less than equal to 9; 'x2' is less than equal to 11 and 'x1', 'x2' is greater than equal to zero.

Here's a summary of our learning in this session, where we have understood:

- The Concept of Linear Programming problems-Formulations
- The Structure and assumptions of Linear Programming problems-Formulations
- The General mathematical model of LP problem
- The LP problem model formulation