1. Introduction

Welcome to the series of E-learning modules on Procurement (with and without shortages), proof under continuity assumptions only.

By the end of this session, you will be able to:

- Understand inventory control models with shortages

Introduction
The inventory models that were discussed before were based on the assumption that shortages and back ordering are not allowed.
As a result, all the EOQ models presented, involved a trade-off between ordering cost and carrying cost.

However, there could be situations in which an economic advantage may be gained by allowing shortages to occur.
One advantage of allowing shortages is to increase the cycle time, and hence spreading the ordering (or set-up) costs over a longer period.

Another advantage of shortage may be seen where the unit value of the inventory and hence the inventory carrying cost is high.
Normally, the benefits due to reduced carrying costs or less number of orders in a planning period are less than the increase in the total inventory costs due to a shortage condition.

2. EOQ Model with Constant Demand and Variable Order Cycle Time

EOQ Model with Constant Demand and Variable Order Cycle Time
For this model, the following characteristics (or inputs) are assumed:

- The inventory system involves one type of item or product
- The demand is known and constant and is resupplied instantaneously
- The inventory is replenished in single delivery for each order
- Lead time (LT) is constant and known, that is replenishment is instantaneous, so that inventory increases by 'Q' units as soon as an order is placed
- Shortages are not allowed. That is, there is always enough inventory on hand to meet the demand
- Purchase price and reorder costs do not vary with the quantity ordered. That is, quantity discount is not available
- Carrying cost per year (as a fraction of product cost) and ordering cost per order are known and constant
- Each item is independent and money cannot be saved by substituting by other items or grouping cost several items into a single order

This model is based on the above assumptions, except that the inventory system runs out of stock for a certain period of time, that is, shortages are allowed.
The cost of a shortage in this case is assumed to be directly proportional to the average number of units short.

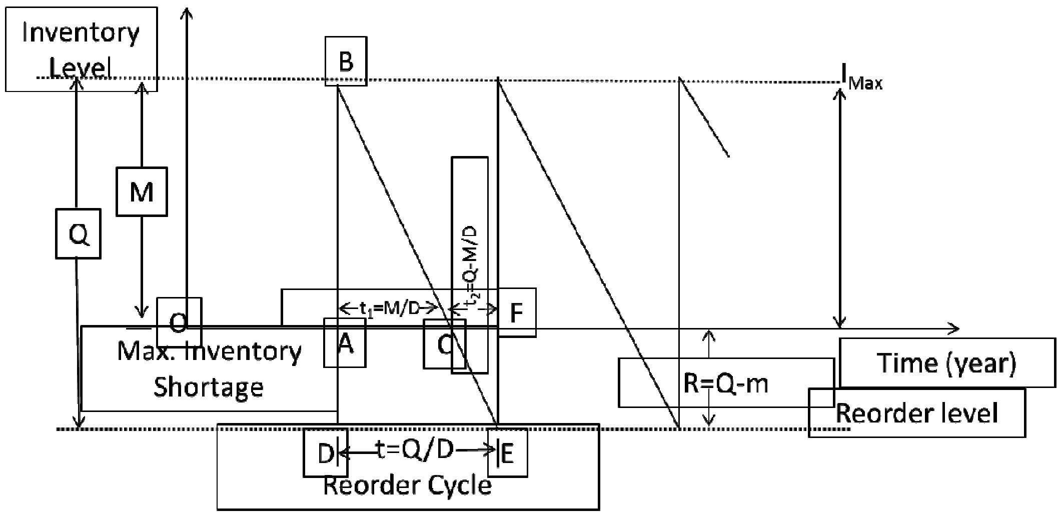
Usually two types of situations occur when an inventory system runs out of stock.
a. Customers are not likely to purchase inventory items, and therefore, any sale that would have resulted is lost
b. Customers wait to receive an order from the supplier and such backorder is filled on stock availability

The backorder cost that is, cost of keeping back log reorders, cost of shipping the items to the customers, loss of goodwill etc depends upon how long a customer waits to receive an order. It is expressed in rupees per unit of time.

In addition to the previously used notations, let
T one is equal to time between the receipt of an order and when the inventory level drops to zero, that is, time when no shortages exist
T two is equal to time during which back order or shortage exists
T is equal to total cycle time. 't' is equal to T one plus T two
R is equal to maximum shortage in units.

The following figure describes the change in the inventory level with time.

Figure 1



Every time the quantity Q (or order size) is received, all shortages equal to amount R are first taken care of. The remaining quantity M is placed in inventory as the surplus will be used to satisfy the demand during the next cycle. Here it may be noted that R units out of Q are always in the shortage list, that is, these are never carried in stock. Thus it yields savings on the inventory carrying cost.

In this inventory system except for the purchase cost C, which will be fixed, all other types of costs will be affected by the decision concerning Q and M.
So we would like to determine optimal value of order quantity, Q star, and optimal stock level, M star along with the optimal shortage level R star.
Thus in this case, we seek to minimize the total variable inventory cost.

TVC is equal to Ordering cost plus Carrying cost plus Shortage cost
Since we want to calculate an optimal value of Q and M, we need to express T one and t in terms of Q and M

Time period in Days is equal to Total units overtime period by Demand in units per day.
Therefore, the time T one when positive inventory level is available is given by T one is equal to M by D
The total cycle time T is given by T is equal to Q by D and the time T two during which shortage incurred is given by T two equal to (Q minus M) by D.

Thus, the average inventory level over the reorder cycle time, T can be determined by dividing the area of the triangle ABC by the cycle time, that is, Average inventory level is equal to Average level over T one plus Average level over T two by T which is equal to (M by two) into T one plus zero point T two the whole divided by T which is equal to M square by two Q

For T one equal to M by D, T equal to Q by D and therefore carrying cost is equal to M square by two Q into C H

Similarly, the average shortage overtime T can be determined by dividing the area under the triangle CFE by the cycle time T, that is, average shortage level (in units) is equal to average level over T one plus average level over T two the whole divided by T which is equal to zero point T one plus (Q minus M by two) into T two, the whole divided by T which is equal to (Q minus M) the whole square by two Q

For T two equal to Q minus M by D and therefore, shortage cost equal to (Q minus M) t whole square by two Q into C S

Hence, the total yearly variable inventory cost is given by TVC in brackets Q coma M, is equal to D by Q into C naught plus M square by two Q into C H plus (Q minus M) the whole square by two Q into C S

Since TVC is the function of two variables Q and M, therefore, in order to determine the optimal order size and the optimal shortage level, R, differentiate the total variable cost function with respect to Q and M; set the two resulting equations equal to zero and solve them simultaneously.

By doing so, we get; Q star is equal to square root of two D C naught by C H into (C H plus C S by C S)
Economic order quantity and M Star is equal to square root of two DC naught by C H into (C S by C H plus C S), Optimal Stock level.

By submitting these values in TVC equation, the minimum so obtained is as follows:
TVC star is equal to square root of two D C naught C H into (C S by C H plus C S)

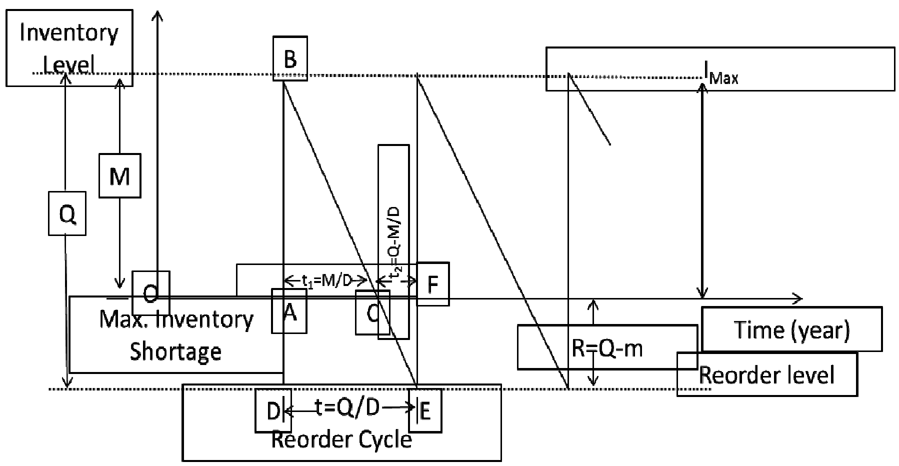
Other Important Formulae
Once we have computed values for Q star and M star, the following can be determined as follows:
1. Optimal shortage level (in units), R star equal to Q star minus M star which is equal to Q star into (C h by C H plus C S)
2. Total cycle time, T is equal to Q star by D which is equal to square root of two DC naught by DC H into (C H plus C S by C S)

Remark:
If in addition to the assumption made for this model, production cost C D per item is given, then TVC will become :
TVC is equal to (D by Q into C naught) plus (M square by two Q into C H) plus (Q minus M the whole square by two Q into C s) plus (D into C d)
Since D into C d equal to constant, therefore optimum value of TVC will remain unaffected.

3. EOQ Model with Constant Demand and Fixed Reorder Cycle Time

EOQ model with constant demand and fixed reorder cycle time
Let the reorder cycle time, T be fixed, that is, the inventory is to be supplied after every time period T. Also, let Q equal to D into T, where D is the demand rate per unit time, Q is the fixed lot size to meet the demand for the period T.

Figure 2



As shown in the figure the amount M(is less than Q) into N is planned to meet the demand during time, T one is equal to M by D
Since the reordering (or set up) cost and time T are constant, therefore the total variable inventory cost (TVC) is given by:
TVC(M) is equal to carrying cost plus shortage cost is equal to M square by two Q into CH plus one by two Q (Q minus M) the whole square into C S

Since the TVC is the function of only M, therefore, the optimal value of M and minimum value of TVC is obtained by differentiating TVC of M with respect to M and then equating it with zero.

On simplifying, we get M is equal to (C S by CH plus C S)into Q which is equal to (C S by CH plus C S) D into t, optimal inventory level by substituting this value of M in TVC equation, the minimum, so obtained is as follows:
TVC star is equal to (CH into C S by CH plus C S) D into t, optimal cost

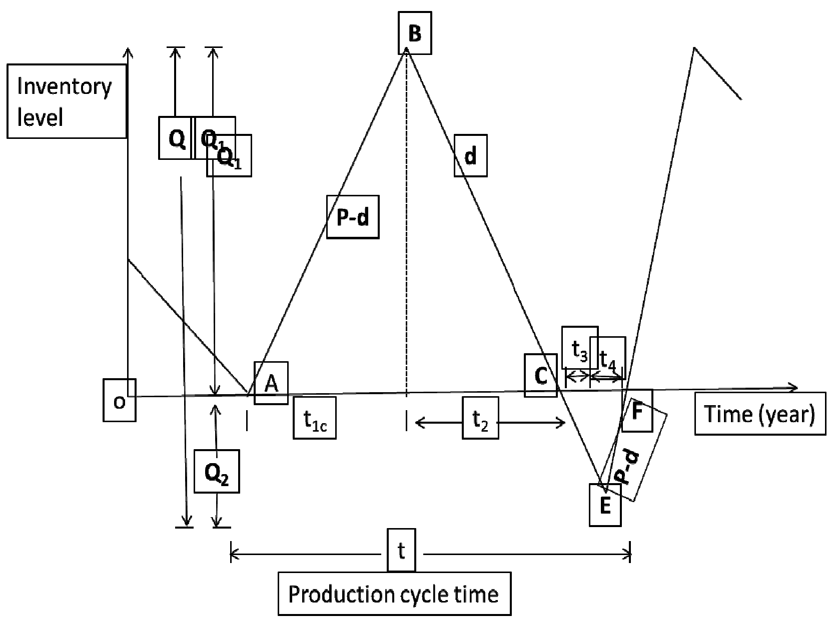
4. EOQ Model with Gradual Supply and Shortage Allowed

EOQ model with gradual supply and shortage allowed
This model is based on the assumption as below except that shortages are allowed.

1. Demand is continuous and at a constant rate
2. During the production run, the production of the item is continuous and at a constant rate until production of quantity (Q) is complete.
3. The rate of receipt (p) of replenishment of inventory (that is items received per unit time) is greater than the usage rate (d) (that is items consumed per unit time)
4. Production runs in order to replenish inventory are made at regular interval
5. Production set-up cost is fixed (independent of quantity produced)

The inventory system is shown in the diagram.

Figure 3



The aim in this situation is to minimize the total yearly variable inventory cost. Total variable cost (TVC) is equal to set-up cost plus carrying cost plus shortage cost.

EOQ Model with Constant Demand and Fixed Reorder Cycle Time
As in the case of this model, the maximum inventory level, say Q ONE, reached at the end of time T one is given by Q ONE is equal (p minus d)T one.

After time T one, the stock Q ONE is used up during T two, thus we have: Q ONE is equal to capital D into T two or small letter D into T two (assuming capital D is equal to small letter D). During time T three, shortage accumulates at the rate of D

Thus the maximum shortage occurred is given by Q TWO is equal to capital D into T three or small letter D into T three (assuming capital D is equal to small d).

After time T three, the production starts and therefore shortage starts reducing at the rate of (p minus d) into T four.
That is, the average inventory and amount of shortage during the production cycle time T are given by:
Average inventory is equal to half of Q ONE into (T one plus T two) by T and average shortage is equal to half of Q TWO into (T three plus T four) by T.

But, production cycle T is equal to T one plus T two plus T three plus T four is equal to Q ONE by p minus d plus Q ONE by d plus Q TWO by D plus Q TWO by (p minus d) is equal to Q ONE into (one by P minus d) plus (one by d) plus Q TWO into (one by d plus one by (p minus d) is equal to p by d into (p minus d) into (Q ONE plus Q TWO).

Now if Q is the lot size, then Q ONE is equal to Q minus Q TWO minus D into T one minus D into T four.
This is equal to Q minus Q TWO minus D into (Q ONE by p minus d plus one by p minus d).
Which is equal to (p minus d by p) into Q minus Q TWO. Q is equal to d into t or Q ONE plus Q TWO is equal to (p minus d by p) into Q. Substituting value of Q ONE plus Q TWO for t (that is, production cycle time) we get,
'T' is equal to p by d into (p minus d) into (p minus d by p) into Q is equal to Q by d.

Hence the expression for TVC can be written as
TVC (Q coma Q ONE coma Q TWO) is equal to D by (Q into C naught) plus half Q ONE into (T one Plus T two) by (T into CH) plus half into Q TWO (T three plus T four) by T into CS.
This is equal to D by Q into C naught plus half Q into P by (P minus d) into (Q ONE square into CH plus Q TWO square into CS)
Or TVC (Q coma Q TWO) is equal to D by Q into C naught plus half Q into (p by p minus d) into [C H into {p minus d by p into Q minus Q TWO} the whole square plus Q TWO square into C s].

The necessary condition for minimum value of TVC is obtained by differentiating TVC partially with respect to Q TWO and Q in the usual manner and then equating them with zero.
On simplifying we get Q star is equal to square root of two D C naught into(C H plus C S) by CH CS into (one minus D by P) is equal to square root of two D into C naught into (C H plus C S) by CH CS into(pd by p minus d)

Since second order partial derivative of TVC with respect to Q TWO and Q are both positive, therefore values of Q and Q TWO, so obtained are optimum values and which minimizes values of TVC.
Q star is equal to square root of two D into C naught by CH into (p by p minus d) into (CH plus CS by CS), the optimal production lot size and
Q TWO star is equal to Q star into (one minus D by p) into (CH by CH plus CS), which is the optimal level of shortage.

Other important formulae:
Production cycle time, T star is equal to Q star by D
Which is equal to one by D into square root of two into D C naught by CH into (p by p minus

d) into (CH plus CS by CS)
This is equal to square root of two into C naught by D into CH into (p by p minus d) into (CH plus CS by CS)

Optimal inventory level, Q ONE star is equal to (p minus d by p) into Q star minus Q TWO star
This is equal to square root of two into D into C naught by CH (i minus d by p) into (CS by CH plus CS).
Total minimum variable inventory cost, TVC star is equal to square root of two into D into C naught CH into (i minus d by p) into (CS by CH plus CS)

REMARKS
a) If p is equal to infinite, then the various results obtained in the model are reduced to the form: Q star is equal to square root of two into D C naught by CH into (CH plus CS by CS)
Q ONE star is equal to square root of two into DC naught by CH into (CS by CS plus CS and TVC star is equal to square root of two into D C naught CH into (Cs by CH plus CS)

b) If CS is equal to infinite, then the results of the model will be the same as TVC star is equal to square root of two DC naught CH into (one minus d by p)

c) If CS is equal to infinite, and then p is equal to infinite then results of this model will be the same as that of TVC is equal to square root of two into D C naught by CH

5. Examples

Let us take a look at a few examples to understand the models explained above:

Example 1

A commodity is to be supplied constant rate of two hundred units per day. Supplies of any amount can be obtained, but each ordering costs fifty rupees. Cost of holding the commodity in inventory is two rupees per unit per day while the delay in the supply of the item induces a penalty of ten rupees per unit per day.

Find the optimal policy (Q, t) where T is the reorder cycle period and Q is the inventory after reorder, what would be the best policy to adopt if the penalty cost becomes infinite?

Solution: From the data given in the problem in usual notations, we have D is equal to two hundred units per day

C naught is equal to fifty rupees per order

CH is equal to two rupees per unit per day; and CS is equal to ten rupees per unit per day.

a) Optimal order quantity: Q star is equal to square root of two into D into C naught by CH into { CH plus CS by CS} which is equal to square root of two into two hundred into fifty by two into {two plus ten by ten} is equal to hundred and nine point five units.

b) Reorder cycle time: t star is equal to Q star by D is equal to hundred and nine point five units by two hundred is equal to zero point five four seven days.

Thus optimal order quantity of hundred and nine point five units must be supplied after every zero point five four seven days.

If the penalty cost CS is equal to infinite, the expression for Q star will become Q star is equal to square root of two into D C naught by CH into { CH plus CS by CS} is equal to square root of two into D C naught by CH is equal to square root of two into two hundred into fifty by two is equal to hundred units

And t star is equal to Q star by D is equal to hundred by two hundred is equal to half day.

Example 2 :

A commodity is to be supplied at a constant rate of twenty five units per day. A penalty cost will be charged at a rate of ten rupees per unit per day, if it is late for missing the schedule delivery date. The cost of carrying the commodity in inventory is sixteen rupees per unit per month. The production process is such that each month (that is thirty days) a batch of items is started and is available for delivery any time after the end of the month. Find the optimal level of inventory at the beginning of each month.

Solution: From the data given in the problem in usual notations, we have: D is equal to twenty five units per day; CH is equal to sixteen rupees by thirty is equal to zero point five three units per day.

CS is equal to ten rupees per unit per day and T is equal to thirty days.

Thus the optimal inventory level is given by M star is equal to CS by CH plus CS into D into T is equal to ten by zero point five three plus ten into twenty five into thirty is equal to seven hundred and twelve units.

Example 3

The demand for an item in a company is eighteen thousand units per year and the company can produce the item at a rate of three per month. The cost of one set-up is five hundred rupees and the holding cost of one unit per month is fifteen paisa. The shortage cost of one unit is two hundred and forty rupees per year. Determine the optimum manufacturing quantity and the number of shortages. Also determine the manufacturing time and the time between set-ups.

Solution:
From the data given in the problem using the usual notations we have: capital
D is equal to small D which is equal to eighteen thousand units per year which is equal to fifteen thousand units per month.
P is equal to three thousand units per month
CH is equal to zero point one five rupee per unit per month.
C naught is equal to five hundred rupees per set-up and CS is equal to two hundred and forty rupees per year or twenty rupees per month.

a) Optimal batch quality Q star is equal to square root of two into D C naught by CH into (p by p minus d) into (CH plus CS by CS) is equal to square root of two into one thousand five hundred into five hundred by zero point one five into (three thousand by three thousand minus one thousand five hundred) into (zero point one five plus twenty by twenty) is equal to four thousand four hundred and eighty nine units

b) Optimal number of shortages Q star is equal to CH by CH plus CS into (one minus d by p) which is equal to zero point one five by zero point one five plus twenty into (one minus one thousand five hundred by three thousand) into four thousand four hundred and eighty nine is equal to seventeen units (approximately)

c) Production time, T one is equal to Q star by P is equal to four thousand four hundred and eighty nine by three thousand is equal to one point five months

d) Production cycle time, T is equal to Q star by P is equal to four thousand four hundred and eighty nine by one thousand five hundred is equal to three months.

Here's a summary of our learning in this session where we have understood the following:

- Implementing a single item inventory control models with shortage