1. Introduction

Welcome to the series of E-learning modules on Assignment Problem and Hungarian Problem (without Proof)

By the end of this session, you will be able to:

- Explain mathematical model of Assignment Problem
- Explain solution methods of Assignment Problems
- Explain variations of the Assignment Problem
- Explain travelling Salesman Problem
- Explain how to solve a typical Assignment Problem

Let us start with an Introduction:
An assignment problem is a particular case of a transportation problem where the sources are assigned and the destinations are tasks.
Furthermore, every source has a supply of one (since each assignee is to be assigned to exactly one task) and every destination has a demand of one (since each task is to be performed by exactly one assignee).
Also, the objective is to minimize the total cost or to maximize the total profit of allocation.

The problem of assignment arises because the resources that is available such as men, machine etc has varying degree of efficiency for performing different activities. Therefore, the cost, profit or time of performing different activities is also different. Thus, the problem is 'how the assignments should be made so as to optimize the given objective'.

Some of the problem where the assignment technique may be useful are assignments of workers to machine, salesman to different sales areas, clerks to various check counters, teacher to classroom, vehicle to routes, contracts to bidders, etc.

Mathematical model of assignment problem:
Given n resources or facilities, n activities or jobs, and effectiveness in terms of cost, profit, time, etc of each resource for each activity , the problem lies in assigning each resource to one and only one activity so that the given measures of effectiveness is optimized.

The data matrix for this problem is shown in the table.

Figure 1

Resources (Workers)	Activities (jobs)	Supply
W_1	C_{11} C_{12} …… C_{1n}	1
W_2	C_{21} C_{22} …… C_{2n}	1
⋮	⋮ ⋮ ⋮	⋮
W_n	C_{n1} C_{n2}…… C_{nm}	1
Demand	1 1 ……. 1	n

It may be noted that this data matrix is the same as the transportation cost matrix except that the supply or availability of each of the resources and the demand at each of the destination is taken to be one.
It is due to this fact that the assignments are made one-to-one basis.

Let xij denote the assignment of facility I to job j such that xij is equal to 1 if facility I is assigned to job j, 0 otherwise.
Then, the mathematical model of the assignment problem can be stated as: minimize Z is equal to summation I is equal to 1 and summation j is equal to 1 cij.xij.
Subject to the constrains, summation xij is equal to 1, for all I (resource availability), summation xij is equal to 1, for all j (activity requirement) and xij is equal to 0 or 1, for all I and j,
Where, cij represents the cost of assignment of resource I to activity j.

From the above discussion, it is clear that the assignment problem is nothing but a variation of the transportation problem with two characteristics:
(1) The cost matrix is square matrix and
(2) The optimal solution for the problem would always be such that there would be only one assignment in a given row or column matrix

Remark: In an assignment problem if a constant is added to or subtracted from every element of any row or column of the given cost matrix, then an assignment that minimizes the total cost in one matrix also minimizes the total cost in the other matrix.

2. Solution Methods of Assignment Problem

Solution methods of assignment problem:
An assignment problem can be solved by the following four methods:

1. Enumeration method
2. Transportation method
3. Simplex method
4. Hungarian method

Enumeration method: In this method, a list of all possible assignments among the given resources that is men, machines, etc and activities that is jobs, sales areas, etc is prepared.
Then an assignment that involves the minimum cost or maximum profit), time or distance is selected.
If two or more assignments have the same minimum cost or maximum profit), time or distance, the problem has multiple optimal solutions.

In general, if an assignment problem involves ‘n’ workers by jobs, then in total there are ‘n’ factorial possible assignments.
For example, for ‘n’ is equal to 5 workers by job problem, we have to evaluate a total of 5 factorial or 120 assignments.
However, when ‘n’ is large, the method is unsuitable for manual calculations.
Hence, this method is suitable only when the value of ‘n’ is small.

Transportation method: Since an assignment problem is a special case of the transportation problem, it can also be solved by transportation methods as already discussed.
However, every basic feasible solution of a general assignment problem that has a square payoff matrix of order ‘n’ should have ‘m’ plus n minus 1 is equal to n plus n minus 1 is equal to 2n minus 1 assignments.

But due to the special structure of this problem, none of this solution can have more than n assignments.
Thus, the assignment problem is inherently degenerate.
In order to remove degeneracy, n minus 1 number of dummy allocations (deltas or epsilons) will be required in order to proceed with the algorithm for solving a transportation problem.

Thus, the problem of degeneracy at each solution makes the transportation method computationally inefficient for solving an assignment problem.

Simplex method: Since each assignment problem can be formulated as a zero or one integer linear programming problem, such a problem can also be solved by the simplex method.

As can be seen in the general mathematical formulation of the assignment problem, there are n into n decision variables and n plus n or 2n equalities.
In particular, for a problem that involves 5 workers/jobs, there will be 25 decision variables and 10 equalities.
This again, is difficult to solve manually.

Hungarian Method:
The Hungarian method (developed by Hungarian mathematician D. konig) of assignment provides us with an efficient method of finding the optimal solution, without having to make a direct comparison of every solution.

It works on the principle of reducing the given cost matrix to a matrix of opportunity costs.
Opportunity costs show the relative penalties associated with assigning a resource to an activity as opposed to making the best or least cost assignment.

If we can reduce the cost matrix to the extent of having at least one zero in each row and column, it will be possible to make optimal assignments (when the opportunity costs are all zero).

3. Variations of the Assignment Problem

Let us now discuss about some of the Variations of the assignment problem:

Multiple optimal solutions: While making an assignment in the reduced assignment matrix, it is possible to have two or more ways to strike off a certain number of zeros, such a situation indicates that there are multiple optimal solutions with the same optimal value of objective function. In such cases the more suitable solution may be considered by the decision-maker.

Maximization case in assignment problem:
There may arise situations when the assignment problems call for maximization of profit, revenue, etc. as the objective function.
Such problems may be solved by converting the given maximization problem into a minimization problem in either of the following two ways:

i. Put a negative sign before each of the payoff element in the assignment table so as to convert the profit values into cost values.
ii. Locate the largest payoff element in the assignment table and then subtract all the elements of the table from the largest element

The transformed assignment problem so obtained can be solved by using the Hungarian method.

Unbalanced Assignment Problem
The Hungarian method of assignment discussed above requires that the number of columns and rows in the assignment matrix be equal.
However, when the given cost matrix is not a square matrix, the assignment problem is called an unbalanced problem.

In such cases dummy row or column are added in the matrix (with zeros as the cost element) in order to make it a square matrix.
For example, when the given cost matrix is of order 4 by 3, a dummy column would be added with zero cost element in that column.
After making the given cost matrix a square matrix, the Hungarian method can be used to solve the problem.

Restrictions on Assignments:
Sometimes it may so happen that a particular resource (say a man or machine) cannot be assigned a particular activity (say territory or job).
In such cases, the cost of performing that particular activity by a particular resource is considered to be a very large (written as M or infinity) so as to prohibit the entry of this pair of resource activity into the final solution.

Travelling Salesman Problem:
The travelling salesman problem may be solved as an assignment problem, with two additional conditions on the choice of assignment. That is how should a travelling salesman travel starting from his home city (the city from where he started) visiting each city only once and returning to the home city, so that the total distance (cost or time) covered is minimum.

For example, given n cities and distances dij (cost cij or time tij) from city I to city j, the salesman starts from city 1, then any permutation of 2, 3 up to n represents is to select an optimal route that is able to achieve the objective of the salesman.
To formulate and solve this problem, let us define xij is equal to 1 if salesman travels from city I to city j or 0 otherwise.
Since each city can be visited only once, we have summation xij is equal to 1 where j is equal to 1, 2 up to n and I is not equal to j.
Again, since the salesman has to leave each city except city n, we have summation xij is equal to 1, I is equal to 1, 2 up to n minus 1; I is not equal to j.

The objective function is then minimize Z is equal to summation I equal to I summation j equal to 1.
Here we do not require dji equal to dij, therefore dij is equal to infinite for I equal to j.
However, all dij's must be non negative, that is dij is greater than or equal to zero and dij plus djk is greater than or equal to djk for all I, j, k.

4. Hungarian Method - Steps

Hungarian method for solving assignment problem:
The Hungarian method (minimization case) can be summarized in the following steps:

Step1: Develop the cost table from the given problem:
If the number of rows are not equal to the number of columns, then as required a dummy row or dummy column must be added.
The cost elements in dummy cells are always zero.

Step 2: Find the opportunity cost table:
Identify the smallest element in each row of the given cost table and then subtract it from each element of that row and in the reduced matrix obtained from step 2a, identify the smallest element in each column and then subtract it from each element of that column.
Each row and column now have at least one zero element.

Step3: Make assignments in the opportunity cost matrix:
The procedure of making assignments is as follows:

a. First round for making assignments

- Identify rows successively from top to bottom until a row with exactly one zero element is found. Make an assignment to this single zero by making a square around it. Then cross off all other zeros in the corresponding column
- Identify columns successively from left to right hand with exactly one zero element that has not been assigned. Make assignments to this single zero by making a square around it and then cross off all other zero elements in the corresponding row

b. Second round for making assignments

- If a row and /or column has two or more unmarked zeros and one cannot be chosen by inspection, then choose the zero cell arbitrarily for assignment
- Repeat steps (a) and (b) successively until one of the following situations arise

Step4: optimality criterion:

a. If all zero elements in the matrix are either marked with square or are crossed and if there is exactly one assignment in each row and column, then it is an optimal solution. The total cost associated with this solution is obtained by adding the original cost figures in the occupied cells

b. If a zero element in a row or column was chosen arbitrarily for assignment in step 4a, there exists an alternative optimal solution.'

c. If there is no assignment in a row or a column, then this implies that the total numbers of assignments are less than the number of row/columns in the square matrix. In such a situation proceed to step5

Step5: Revise the opportunity cost matrix:

Draw a set of horizontal and vertical lines to cover all the zeros in the revised cost matrix obtained from step 3, by using the following procedure:

a. For each row in which no assignment was made , mark a tick

b. Examine the marked rows. If any zero element occurs in those rows, mark a tick to the respective columns containing those zeros

c. Examine marked columns. If any assigned zero element occurs in those columns mark a tick to the respective rows containing those assigned zeros

d. Repeat those process until no rows or columns can be marked

e. Draw a straight line through each marked column and each un-marked row

If the number of lines drawn or total assignments is equal to the number of rows or columns, the current solution is the optimal solution, otherwise go to step6.

Step 6: Develop the new revised opportunity cost matrix:

a. From among the cells not covered by any line, choose the smallest element. Call this value k

b. Subtract k from every element in the cell, that is not covered by a line

c. Add k to every element in the cell covered by the two lines, which is intersection of two lines

d. Elements in cells covered by one line remain unchanged

Step 7: Repeat steps:

Repeat steps 3 to 6 until an optimal solution is obtained

5. Examples

Let us take an example to understand the assignment problem using the Hungarian method.

Example 1:

A department of a company has five employees with five jobs to be performed. The time in hours that each man takes to perform each job is given in the effectiveness matrix below.

How should the jobs be allocated, one per employee, so as to minimize the total man hours?

Figure 2

Jobs / Employees	I	II	III	IV	V
A	10	5	13	15	16
B	3	9	18	13	6
C	10	7	2	2	2
D	7	11	9	7	12
E	7	9	10	4	12

Row represents the employee A,B,C,D & E & column represents the Jobs 1,2,3,4 & 5 respectively.

The values in each box represents the time (in hours) that each man takes to perform each job.

Solution:

By applying **step 2** of the algorithm, let us get the reduced opportunity time matrix by taking the minimum element in the rows and subtracting this element from all elements in their respective row. The reduced matrix is shown in the table

Figure 3

Jobs / Employees	I	II	III	IV	V
A	5	0	8	10	11
B	0	6	15	10	3
C	8	5	0	0	0
D	0	4	2	0	5
E	3	5	6	0	8

Step 3 and 4:

Figure 4

Employees \ Jobs	I	II	III	IV	V
A	5	$\boxed{0}$	8	10	11
B	$\boxed{0}$	6	15	10	3
C	8	5	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$
D	$\cancel{0}$	4	2	$\cancel{0}$	5
E	3	5	6	$\boxed{0}$	8

We examine all the rows starting from A, one-by-one, until a row containing only single zero element is located. Here rows A, B and E have only one zero element in the cells (A,2), (B,1) AND (E,4).
Assignment is made in these cells. All zeros in the assigned columns are now crossed off as shown in table.
We now examine each column starting from column 1. There is one zero in column 3, cell (C,3). Assignment is made in this cell. Thus cell (C,5) is crossed off. All zeros in the table are now either assigned or crossed off as shown in the table. The solution is not optimal because only four assignments are made.

Figure 5

Employees \ Jobs	I	II	III	IV	V	
A	5	$\boxed{0}$	8	10	11	
B	$\boxed{0}$	6	15	10	3	✓
C	8	5	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	
D	$\cancel{0}$	4	2	$\cancel{0}$	5	✓
E	3	5	6	$\boxed{0}$	8	✓
	✓			✓		

Step 5: Cover the zeros with minimum number of lines (which is equal to 4) as explained below:

1. Mark or tick row D since it has no assignment
2. Tick columns 1 and 4 since row D has zero element in these columns
3. Tick rows B and E since columns 1 and 4 have an assignment in rows B and E, respectively

4. Since no other rows or columns can be marked, draw straight lines through the unmarked rows A and C and the marked column 1 and 4, as shown in the table

Figure 6

Employees \ Jobs	I	II	III	IV	V
A	5	$\boxed{0}$	8	10	11
B	$\boxed{0}$	6	15	10	3
C	8	5	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$
D	$\cancel{0}$	4	2	$\cancel{0}$	5
E	3	5	6	$\boxed{0}$	8

Step 6:

Figure 7

Employees \ Jobs	I	II	III	IV	V
A	7	$\boxed{0}$	8	12	11
B	$\boxed{0}$	4	13	10	1
C	10	5	0	2	$\boxed{0}$
D	0	2	$\boxed{0}$	0	3
E	3	3	4	$\boxed{0}$	6

1. Develop the new revised table by selecting the smallest element among all uncovered elements by the lines in table that is 2

2. Subtract 'k' equal to 2 from uncovered elements including itself and add it to elements 5, 10, 8 and 0 in cells (A, 1), (A,4), (C,1) and (C,4) respectively, which lie at the intersection of two lines. Another revised table, so obtained, is shown in this table

Step 7:
Repeat steps 3 to 6 to find a new solution.

Figure 8

Job	Employees	Time (Hr)
A	II	5
B	I	3
C	V	2
D	III	9
E	IV	4
TOTAL		23

The new assignment is shown in this table. Since the assignments (which is equal to 5) equals the number of rows or columns, the solution is optimal. The pattern of assignments among jobs and employees with their respective time is given in the table.

Example 2:
A computer center has three expert programmers. The center wants three application programmes to be developed. The head of the computer center, after carefully studying the programmes to be developed, estimates the computer time in minutes required by the experts for the application programmes as follows:

Row represents the 3 expert programmer 1,2 & 3 and column represents the application programme A,B & C to be developed.

Figure 9

	A	B	C
1	120	100	80
2	80	90	110
3	110	140	120

Values on the table represent the computer time in minutes required by the experts for the application programmes.
Assign the programmers to the programmes in such a way that the total computer time is minimum.

Solution:
The Hungarian method is used to obtain an optimal solution

Figure 10

	A	B	C
1	40	20	0
2	0	10	30
3	0	30	10

Step 1 and 2:

1. The minimum time element in rows 1, 2 and 3 is 80, 80 and 110 respectively. Subtract these elements from all elements in their respective row. The reduced time matrix is shown in the table.

2. In the reduced table the minimum time element in columns A, B and C is zero, 10 and zero respectively. Subtract these elements from all element s in their respective column in order to get the reduced time matrix as shown in the table

Figure 11

	A	B	C
1	40	10	$\boxed{0}$
2	$\cancel{0}$	0	30
3	$\boxed{0}$	20	10

Step 3:

1. Examine all the rows starting from the first, one-by-one, until a row containing only single zero element is located. Here rows 1 and 3 have only one zero in the cells (1,C) and (3, A), respectively. We assign these with zeros. All zeros in the assigned column are crossed off as shown in the table

2. We now examine each column starting from A in the table. There is one zero in column B in the cell (2, B). assign this cell as shown in the table

Figure 12

	A	B	C
1	40	10	$\boxed{0}$
2	$\cancel{0}$	$\boxed{0}$	30
3	$\boxed{0}$	20	10

3. Since the number of assignments is equal to 3 which equals the number of rows, the optimal solution is obtained. The pattern of the assignments among programmers and programmes with their respective time is given in this table

Figure 13

Programmer	Programme	Time (min)
1	C	80
2	B	90
3	A	110
Total		**280**

Here's a summary of our learning in this session, where we have understood:

- The mathematical model of Assignment Problem
- The solution methods of Assignment Problems
- The variations of the Assignment Problem
- The travelling Salesman Problem
- The typical Assignment Problems