1. Introduction

Welcome to the series of E-learning modules on U-V method of solving transportation problem (without proof).

At the end of this session, you will be able to:

- Derive the optimal solution by using the method of modified distribution

Let us start with an introduction:
Once an initial solution is available, the next step is to check its optimality.
An optimal solution is one in which there is no opportunity cost. That is, there is no other set of transportation routes or allocation that will reduce the total transportation cost.

Thus, we have to evaluate each unoccupied cell which represent unused route, in the transportation table in terms of opportunity cost.
The unoccupied cell with the largest negative opportunity cost is selected to include in the new set of transportation routes. This is also known as incoming variable.

The outgoing variable in the current solution is the occupied cell which is basic variable in the unique closed path or loop whose allocation will become zero first, as more units are allocated to the unoccupied cell with the largest negative opportunity cost. Such an exchange reduces total transportation cost.
The process is continued until there is no negative opportunity cost. That is the current solution can be improved further. This is the optimal solution.
Here we shall discuss an efficient technique called the modified distribution (MODI) method also called as the u-v method which helps in comparing the relative advantage of alternative allocations for all unoccupied cells simultaneously.

The MODI method is based on the concept of duality.
Dual of transportation model:
For a basic feasible solution if we associate numbers (also called dual variables or multipliers) ‘ui’ and ‘vj’ with row ‘i’ where ‘i’ equal to 1, 2, 3 up to ‘m’ and column ‘j’ where ‘j’ equal to 1, 2, 3 up to ‘n’ of the transportation table respectively, then ‘ui’ and ‘vj’ must satisfy the equation ‘ui’ plus ‘vj’ is equal to ‘cij’ for each occupied cell i, j.

These equations yield m plus n minus 1, equation m plus n is unknown dual variables. The values of these variables can be determined from the above relationship by assigning arbitrarily zero value to any one of these variables and then the value is of the remaining m plus n minus 1 variable can be obtained algebraically.

Once the values ‘ui’ and ‘vi’ have been determined, evaluation in terms of opportunity cost of each unoccupied cell called non basic variable or unused route is done using the equation:
‘drs’ is equal to ‘crs’ minus ‘ur’ plus ‘vs’, for each unoccupied cell r, s.

2. Economic Interpretation of ui's and vj's

Economic interpretation of ui's and vj's.
The value of each variable 'ui' measure the comparative advantage of either the location or the value of a unit of capacity at the supply center i and, therefore, may be termed as location rent.

Similarly, the value of each variable 'vj' measures the comparative advantage of an additional unit of commodity transported to demand centre 'j' and, therefore, may be termed as market price.

Illustration:
The concept of duality in transportation problem is applied in the following manner

Figure 1

Production facility	Production capacity/ week (in 100's)	Warehouses	Demand/ week (in 100's)
S_1	7	D_1	5
S_2	9	D_2	8
S_3	18	D_3	7
		D_4	14

A company has three production facilities S1, S2 and S3 with production capacity of 7, 9 and 18 units in 100's per week of a product respectively. These units are to be shipped to four warehouses D1, D2, D3 and D4 with requirement of 5, 8, 7 and 14 in 100's per week, respectively.

The transportation costs in Rupees per unit between factories to warehouses are given in the table.

Figure 2

From \ To	D_1	D_2	D_3	D_4	Capacity
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	**34**

In the table, we have taken the sources of supply in the rows and the units of demand in the columns. The values of cost from the source to the destination is indicated
Second row shows the transportation cost from production facility s1 to warehouse d1which is 19, to d2 is 30, to d3 is 50 & to d4 is 10.
Similarly, third & fourth row shows the transportation cost from production facility s2 & s3 to warehouse d2, d3 & d4 respectively.
Last row & last column indicates the demand & production capacity respectively where total comes to 34.

In the table we have m equal to 3 rows and n equal to 4 columns.
Let u1, u2 and u3 be dual variables corresponding to each of the supply constraint in that order.
Similarly, v1, v2, v3 and v4 be dual variables corresponding to each of demand constraint in that order.

Figure 3

	D_1	D_2	D_3	D_4	Capacity	u_i
S_1	19	30	50	10	7	u_1
S_2	70	30	40	60	9	u_2
S_3	40	8	70	20	18	u_3
Demand	5	8	7	14	34	
v_j	v_1	v_2	v_3	v_4		

The dual problem then becomes as shown in the table where we have added a column and row indicating the dual variable and is written as maximize Z is equal to (7u1 plus 9u2 plus 18u3) plus (5v1 plus 8v2 plus 7v3 plus 14v4).

subject to the constraints, u1 plus v1 is less than equal to 19, u1 plus v2 is less than equal to 30, u1 plus v3 is less than equal to 50, u1 plus v4 is less than equal to 10, u2 plus v1 is less than equal to 70, u2 plus v2 is less than equal to 30, u2 plus v3 is less than equal to 40, u2 plus v4 is less than equal to 60, u3 plus v1 is less than equal to 40, u3 plus v2 is less than equal to 8, u3 plus v3 is less than equal to 70, u3 plus v4 is less than equal to 20 and ui, vj are unrestricted in sign for all i and j.

For interpreting the data, let us consider the constraints u1 plus v1 is less than equal to 19 or v1 is less than equal to 19 minus u1.
This represents the delivered market value of the commodity at destination D1 which should be less than or equal to the unit cost of transportation from S1 to D1 minus the per unit value commodity at D1.

A similar interpretation Can also be given for other constraints.

Now, the optimal values of dual variables can be obtained either by solving this linear programming problem or by reading values of these variables from the transportation table that contains the optimal solution.

It can be verified that the total transportation cost at optimal solution obtained by the MODI method would be the same as obtained by putting values of ui's and vj's from optimal transportation table in the dual objective function: maximize Z is equal to summation 'ai''ui' plus summation of 'bj''vj'.

3. MODI Method – Steps 1 to 4

Steps of MODI method that is transportation algorithm is discussed below:
The steps to evaluate unoccupied cells are as follows:
Step1: For an initial basic feasible solution with m plus n minus 1 occupied cells, calculate ui and vj for rows and columns.
The initial solution can be obtained by any one of the three methods discussed earlier.

To start with, any one of ui's or vj's is assigned the value zero to a particular ui or vj where there are maximum number of allocation in a row or a column respectively, as this will reduce the considerably arithmetic work.
Then complete the calculations of ui's and vj's for other rows and columns by using the relation cij is equal to ui plus vj, for all occupied cells i, j.

Step 2: For unoccupied cells, calculate the opportunity cost that is,the difference that indicates the per unit cost reduction that can be achieved by an allocation in the unoccupied cell.
Do this by using the relationship dij is equal to cij minus ui plus vj, for all i and j.

Step 3: Examine sign of each dij

i. If dij is greater than 0, then the current basic feasible solution is optimal
ii. If dij is equal to 0,then the current basic feasible solution will remain unaffected but an alternative solution exists
iii. If one or more dij is less than 0, then an improved solution can be obtained by entering unoccupied cell i, j in the basis.
An unoccupied cell having the largest negative value of dij is chosen for entering into the solution mix that is, new transportation schedule.

Step 4:

- Construct a closed path or loop for the unoccupied cell with largest negative opportunity cost
- Start the closed path with the selected unoccupied cell and mark a plus sign in this cell
- Trace a path along the rows or columns to an occupied cell, mark the corner with a minus sign and continue down the column or row to an occupied cell
- Then mark the corner with plus sign and minus sign alternatively
- Close the path back to the selected unoccupied cell

4. MODI Method – Steps 5 to 7 & Remarks

Step 5:

- Select the smallest quantity amongst the cells marked with minus sign on the corners of closed loop
- Allocate this value to the selected unoccupied cell and add it to other occupied cells marked with plus signs
- Now subtract this from the occupied cells marked with minus sign

Step 6:
Obtain a new improved solution by allocating units to the unoccupied cell according to step 5 and calculate the new total transportation cost.
Step 7:
Further test the revised solution for optimality. The procedure terminates when all dij is greater than equal to 0 for unoccupied cells.

Remarks:

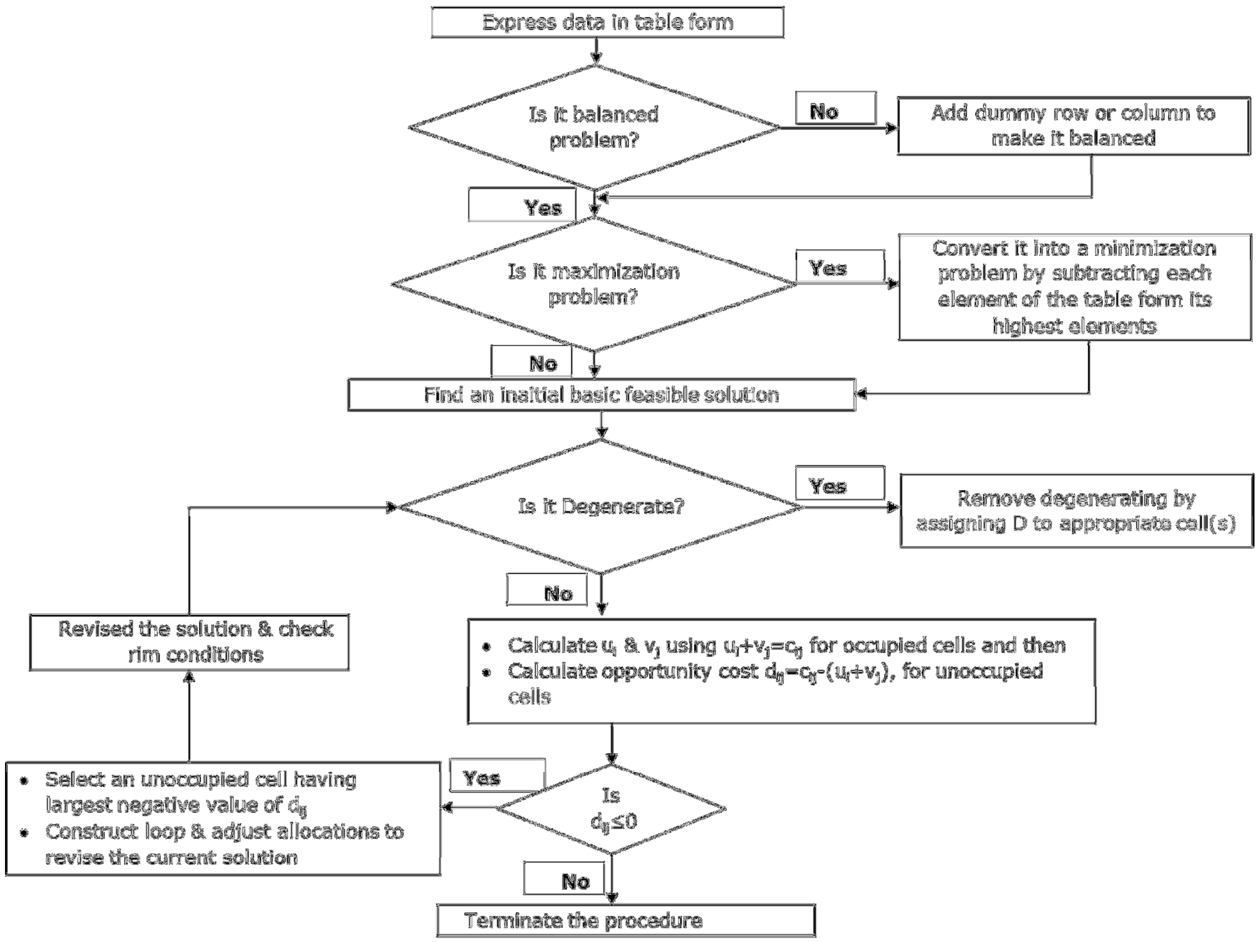
1. The closed-loop (path) starts and ends at the selected unoccupied cell.
   It consists of successive horizontal and vertical (connected) lines whose end points must be occupied cells, except for an end point associated with entering unoccupied cell.
   This means that every corner element of the loop must be an occupied cell

It is immaterial whether the loop is traced in a clockwise or anti-clockwise directions and whether it starts, up, down, right or left but never diagonally.
However, for a given solution only one loop can be constructed for each unoccupied cell.

2. There can only be one plus sign and only one minus sign in any given row or column
3. The closed path indicates changes involved in reallocating the shipments

The steps of MODI method for solving a transportation problem can also be described by the flow chart as shown.

Figure 4



Express data in table form
Is it balanced problem?
No
Add dummy row or column to make it balanced
Yes
Is it maximization problem?
Yes
Convert it into a minimization problem by subtracting each element of the table form its highest elements
No
Find an inaitial basic feasible solution
Is it Degenerate?
Yes
Remove degenerating by assigning D to appropriate cell(s)
No
Revised the solution & check rim conditions
Calculate u_i & v_j using $u_i+v_j=c_{ij}$ for occupied cells and then
Calculate opportunity cost $d_{ij}=c_{ij}-(u_i+v_j)$, for unoccupied cells
Select an unoccupied cell having largest negative value of d_{ij}
Construct loop & adjust allocations to revise the current solution
Yes
Is $d_{ij} \leq 0$
No
Terminate the procedure

5. Illustrative Example

Let us take an example and understand the concept in detail.
A company has three production facilities S1, S2 and S3 with production capacity of 7, 9 and 18 units (in 100's) per week of a product respectively. These units are to be shipped to four warehouses D1, D2, D3 and D4 with requirement of 5, 8, 7 and 14 (in 100's) per week, respectively.

Figure 5

Production facility	Production capacity/ week (in 100's)	Warehouses	Demand/ week (in 100's)
S_1	7	D_1	5
S_2	9	D_2	8
S_3	18	D_3	7
		D_4	14

The transportation costs in Rupees per unit between factories to warehouses are given in the table.

Figure 6

From \ To	D_1	D_2	D_3	D_4	Capacity
s_1	19	30	50	10	7
s_2	70	30	40	60	9
s_3	40	8	70	20	18
Demand	5	8	7	14	**34**

In the table, we have taken the sources of supply in the rows and the units of demand in the columns. The values of cost from the source to the destination is indicated
Second row shows the transportation cost from production facility s1 to warehouse d1which is 19, to d2 is 30, to d3 is 50 & to d4 is 10.
Similarly, third & fourth row shows the transportation cost from production facility s2 & s3 to warehouse d2, d3 & d4 respectively.
Last row & last column indicates the demand & production capacity respectively where total comes to 34.
Find the optimal solution using MODI method.

Solution:
Figure 7

	D$_1$	D$_2$	D$_3$	D$_4$	Capacity	Row Difference				
S$_1$	19	30	50	10	7	9	9	40	-	-
S$_2$	70	30	40	60	9	10	20	10	10	10
S$_3$	40	8 [8]	70	20	18	12	20	20	-	-
Demand	5	8	7	14	**34**					
Column Difference	21	22	10	10						
	21	-	10	10						
	-	-	10	10						
	-	-	10	60						
	-	-	10	-						

1. We apply the vogel's approximation method to obtain an initial basic feasible solution, as shown in the table the differences (penalty costs) for each row and column has been calculated. In the first round, the maximum penalty, that is twenty two, occurs in column D2. The cell (S3, D2) having the least transportation cost eight is chosen for allocation. The maximum possible allocation in this cell is eight and it satisfies demand in column D2. Adjust the supply of S3 from eighteen to ten. That is,18 minus 8 is equal to 10.

Figure 8

	D$_1$	D$_2$	D$_3$	D$_4$	Capacity	Row Difference				
S$_1$	19	30	50	10 [2]	7	9	9	40	-	-
S$_2$	70	30	40	60	9	10	20	10	10	10
S$_3$	40	8 [8]	70	20 [10]	18	12	20	20	-	-
Demand	5	8	7	14	**34**					
Column Difference	21	22	10	10						
	21	-	10	10						
	-	-	10	10						
	-	-	10	60						
	-	-	10	-						

The new row and column penalties are calculated except column D2 because its demand has been satisfied. The second round allocation is made in column D1 with target penalty, 21, the same way as in the first round as shown in the cell S1, D1 of the table. In the third round, the maximum penalty 20 occurs at two places, row S1 and S3. The maximum possible allocation can be made in both the rows at the cells S1, D4 and S3, D4 having least transportation costs 10 and 20 respectively as shown in Table.

The process is continued with new allocations till a complete solution is obtained. The initial solution using Vogel's Approximation Method is shown in the table.

Figure 9

	D_1	D_2	D_3	D_4	Capacity	Row Difference				
S_1	19 [5]	30	50	10 [2]	7	9	9	40	-	-
S_2	70	30	40 [7]	60 [2]	9	10	20	10	10	10
S_3	40	8 [8]	70	20 [10]	18	12	20	20	-	-
Demand	5	8	7	14	34					
Column Difference	21	22	10	10						
	21	-	10	10						
	-	-	10	10						
	-	-	10	60						
	-	-	10	-						

The total numbers of occupied cells are m plus n minus 1 is equal to 3 plus 4 minus 1 is equal to 6 and the initial solution is non-degenerate.
The total transportation cost associated with this method is calculated as:
Total Cost equal to 5 into 19 plus 8 into 8 plus 7 into 40 plus 2 into 10 plus 2 into 60 plus 10 into 20 is equal to Rupees seven hundred seventy nine.

In order to calculate the values of ui's where i is equal to 1, 2, 3, and vj's where j is equal to 1, 2, 3, and 4 for each occupied cell, we arbitrarily assign v4 is equal to 0 in order to simplify calculations.
Given v4 is equal to 0; then, u1, u2, and u3 can be easily computed by using the relation cij is equal to ui plus vj for occupied cells.

The calculations are as shown follows:
C14 is equal to u3 plus v4 or 20 is equal to u3 plus zero or u3 is equal to 20, c24 is equal to u2 plus v4 or 60 is equal to u2 plus 0 or u2 is equal to 60, similarly c14 is equal to u1 plus v4 or 10 is equal to u1 plus 0 or u1 is equal to 10.

Given u1, u2 and u3, value of v1, v2, v3 and v4 can also be calculated as shown below:

C11 is equal to u1 plus v1 or 19 is equal to 10 plus v1 or v1 is equal to 9, C23 is equal to u2 plus v3 or 40 is equal to 60 plus v3 or v3 is equal to minus 20, C32 is equal to u3 plus v2 or 8 is equal to 20 plus v2 or v2 is equal to minus 12.

The opportunity cost for each of the occupied cell is determined by using the relation dij is equal to cij minus ui plus vj and is shown below,
d12 is equal to C12 minus u1 plus v2 is equal to 30 minus 10minus 12 is equal to 32
d13 is equal to c13 minus u1 plus v3 is equal to 50 minus 10 minus 20 is equal to 60
d21 is equal to c21 minus u2 plus v1 is equal to 70 minus 60 plus 9)is equal to 1
d22 is equal to c22 minus u2 plus v2 is equal to 30 minus 60 minus 12 is equal to minus 18
d31 is equal to c31 minus u3 plus v1 is equal to 40 minus 20 plus 9 is equal to 11 and
d33 is equal to c33 minus u3 plus v3 is equal to 70 minus 20 minus 20 is equal to 70.

According to the optimality criterion for cost minimizing transportation problem, the current solution is not optimal, since the opportunity cost of the unoccupied cells are not all zero or positive.
The value of d22 is equal to minus 18 in cell S2, D2 is indicating that the total transportation cost can be reduced in the multiple of 18 by shifting an allocation to this cell.

Figure 10

	D_1	D_2	D_3	D_4	Capacity	u_i
S_1	19	30	50	10	7	$u_1=10$
S_2	70	30 (+)0	40	60 2 (-)	9	$u_2=60$
S_3	40	8 (-) 8	70	20 10 (+)	18	$u_3=20$
Demand	5	8	7	14	34	
v_j	$v_1=9$	$v_2=-12$	$v_3=-20$	$v_4=0$		

A closed loop path is traced along row S2 to an occupied cell S3, D2. A plus sign is placed in cell S2, D2 and minus sign in cell S3,D2. Now take a right angle turn and locate an occupied cell in column D4. An occupied cell S3, D4 exists at row S3 and a plus sign is placed in this cell. Continue this process and complete the closed path. The occupied cell S2, D3 must be bypassed otherwise they will violate the rule of constructing closed path.

Figure 11

	D_1	D_2	D_3	D_4	Capacity	u_i
S_1	19 5	30	50	10 2	7	$u_1=10$
S_2	70	30 (+)2	40 7	60 0 (-)	9	$u_2=60$
S_3	40	8 (-) 6	70	20 12 (+)	18	$u_3=20$
Demand	5	8	7	14	34	
v_j	$v_1=9$	$v_2=-12$	$v_3=-20$	$v_4=0$		

In order to maintain feasibility, examine the occupied cells with minus sign at the corners of the closed loop and select the one that has the smallest allocation. This determines the maximum number of units that can be shifted along the closed path. The minus signs are in cells S3, D2 and S2, D4.
The cell S2, D4 is selected because it has the smaller allocation that is 2. The value of this allocation is then added to cell S2, D2 and S3, D4, which carry plus signs. The same value is subtracted from cells S2, D4 and S3, D2 because they carry minus signs.

The revised solution is shown in the table below. The total transportation cost associated with this solution is 5 into 19 plus 2 into 10 plus 2 into 30 plus 7 into 40 plus 6 into 8 plus 12 into 20 is equal to Rupees seven hundred forty three.

Test the optimality of the revised solution once again in the same way as discussed in earlier steps.
The values of ui's , vj's and dij's are shown in the table.
Since each of dij's is positive, therefore, the current basic feasible solution is optimal with a minimum total transportation cost of rupees seven hundred forty three.

Here's a summary of our learning in this session, where we have understood:

- To drive the optimal solution by using the modified distribution method for a transportation problem