1. Introduction and Unbalanced Supply & Demand

Welcome to the series of E-learning modules on Degenerate transportation problem.

By the end of this session, you will be able to understand:

- Variation in transportation problem
- Maximization in transportation problem
- Trans-shipment problem

Let us start with an introduction on degenerate transportation problem:
While solving a transportation problem we often come across various variations like unbalanced supply and demand, degeneracy and its resolution, alternative optimal solutions, prohibited transportation routes and trans-shipment problem. Let us now discuss each of these variations in detail:

Unbalanced supply and demand:
For a feasible solution to exist, it is necessary that the total supply must equal the total demand.
That is, total supply equal to total demand is expressed as summation ai is equal to summation bj.
But a situation may arise when the total available supply is not equal to the total demand.

The following two cases may arise:

a. **If the total supply exceeds the total demand,** then an additional column called a dummy demand centre can be added to the transportation table in order to absorb the excess supply. The unit transportation cost for the cells in this column is set equal to zero because these represent product items that are not being made and not being sent

b. **If the total demand exceeds the total supply**, a dummy row called a dummy supply centre can be added to the transportation table to account for the excess demand quality. The unit transportation here also for the cells in the dummy row is set equal to zero

Let us take an example:

A company has received a contract to supply gravel to three new construction projects located in towns A, B and C.
The construction engineers have estimated that the required amounts of gravel which will be needed at these construction projects are shown below:

Figure 1

Project Location	Weekly requirement (Truck loads)
A	72
B	102
C	41

First column indicates the project locations A,B & C whereas second column indicates the weekly requirement of Town A, B & C respectively.

The company has 3 gravel pits located in town X, Y and Z. the gravel required by the construction projects can be supplied by three pits. The amount of gravel that can be supplied by each pit is as follows:

Figure 2

Plant	X	Y	Z
Amount available (truck loads)	76	82	77

First row indicates the Plant X, Y & Z whereas second row indicates the amount of truck loads available from plant X, Y & Z respectively.

The company has computed the delivery cost from each pit to each project site. These costs (in Rupees) are shown in the following table:

Figure 3

Project Location / PIT	A	B	C
X	4	8	8
Y	16	24	16
Z	8	16	24

Second row represents the delivery cost from PIT X, to location A, B & C which is 4, 8 & 8 respectively. Similarly third & forth row represents the cost from PIT Y & Z to location A, B & C respectively.
Schedule the shipment from each PIT to each project in such a manner that it minimizes the total transportation cost within the constraints imposed by PIT capacities and project requirements. Also find the minimum cost.

Solution:

Figure 4

PIT \ Project Location	A	B	C	D_{Excess}	Supply
X	4	8 [35]	8 [41]	0	76
Y	16	24 [62]	16	0 [20]	82
Z	8 [72]	16 [5]	24	0	77
Demand	72	102	41	20	235

The total plant availability of 235 truckloads exceeds total requirement of 215 truckloads by 20 truckloads. The excess truck load capacity of 20 is handled by adding a dummy project location (column) D excess with a requirement equal to 20. We use zero unit transportation costs to the dummy project location. The modified transportation table is shown as below where the initial solution is obtained using Vogel's Approximation method. It may be noted that 20 units are allocated from pit X to dummy project location D. This means pit X is short by 20 units.

Figure 5

PIT \ Project Location	A	B	C	D_{Excess}	Supply	u_i
X	4	8 [35] (+)	8 [41] (-)	0	76	$U_1=8$
Y	16	24 (-) [62]	16 (+) [-8]	0 [20]	82	$U_2=24$
Z	8 [72]	16 [5]	24	0	77	$U_3=16$
Demand	72	102	41	20	235	
v_j	$V_1=-8$	$V_2=0$	$V_3=0$	$V_4=-24$		

Now in order to apply optimality test calculate the ui's and vj's corresponding to rows and columns respectively, in the same way as discussed before. These values are shown in the table below where we have calculated the value of ui and vj, we get the ui values as 8, 24 and 16 and the v_j values as minus 8, 0, 0 and minus 24.

In the calculations we have seen that all opportunity costs dij's are not positive, the current solutions is not optimal.

Thus, the unoccupied cell x, c, where d23 is equal to minus 8 must enter into the basis and cell W, C must leave the basis, as shown by the closed path.

The new solution is shown in the next table where we can see that all the opportunity costs dij's are non-negative, the current solution is optimal.

Figure 6

Project Location / PIT	A	B	C	D_{Excess}	Supply	u_i
X	4 +4	8 76	8 +8	0 +16	76	$U_1=8$
Y	16 0	24 21	16 41	0 20	82	$U_2=24$
Z	8 72	16 5	24 +16	0 +8	77	$U_3=16$
Demand	72	102	41	20	235	
v_j	$V_1=-8$	$V_2=0$	$V_3=0$	$V_4=-24$		

The total minimum transportation cost associated with this solution is total cost is equal to 8 into 76 plus 24 into 21 plus 16 into 41 plus 0 into 20 plus 8 into 72 plus 16 into 5 is equal to RS two thousand four hundred twenty four.

2. Degeneracy and It's Resolution - (Case 1)

Degeneracy and its resolution
A basic feasible solution for the general transportation problem must consist of exactly m plus n minus 1 positive allocation in independent positions in the transportation table. A solution will only be called degenerate if the number of occupied cells is less than the required number, m plus n minus 1.

In such cases, the current solution cannot be improved upon because it is not possible to draw a closed path for every occupied cell.
Also the values of dual variables ui and vj, that are used to test the optimality cannot be computed.
Thus, we need to remove the degeneracy in order to improve the given solution.

The degeneracy in the transportation problems may occur at two stages:
a) When obtaining an initial basic feasible solution we may have less than m plus n minus 1 allocations
b) At any stage while moving towards optimal solution. This happens when two or more occupied cells with the same minimum allocation are simultaneously unoccupied

Case 1: Degeneracy at the initial solution:
To resolve degeneracy at the initial solution, we proceed by allocating a very small quantity close to zero to one or more (if needed) unoccupied cells so as to get m plus n minus 1 number of occupied cells.
This amount is denoted by a greek letter epsilon or delta. This quantity would neither affect the total cost nor the supply and demand values.

In a minimization transportation problem, it is better to allocate delta to unoccupied cells that have lowest transportation costs, whereas in maximization problems, it should be allocated to a cell that has a high payoff value.
In some cases, delta must be added in one of those unoccupied cells that uniquely makes possible the determination of ui and vj.

The quantity delta is considered to be so small that if it is transferred to an occupied cell it does not change the quantity of allocation.
That is, xij plus delta is equal to xij, Delta minus delta is equal to 0, 0 plus delta is equal to delta, Minus delta is equal to xij, delta plus delta is equal to delta and k into delta is equal to delta.

It is also obvious that delta does not affect the total transportation cost of the allocation. Hence the quantity delta is said to evaluate unoccupied cells and to reduce the number of improvement cycles necessary to reach an optimal solution.

Once the purpose is over, delta can be removed from the transportation table.

Let us take an example to understand the concept of degeneracy at the initial solution: A manufacture wants to ship 22 loads of his product as shown below. The matrix gives the kilometers from sources of supply to the destinations.

Figure 7

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	5	8	6	6	3	8
S_2	4	7	7	6	5	5
S_3	8	4	6	6	4	9
Demand	4	4	5	4	8	25 \ 22

In the table, second row represents the number of kilometer travelled from source of supply s1 to destination d1, d2, d3, d4 & d5. Similarly third & fourth row represents the number of kilometer travelled from s2 & s3 to different destinations respectively.
Last row & column shows the demand & supply where total comes to 25 & 22 respectively.
The shipping cost is Rs10 per load per km. What shipping schedule should be used in order to minimize the total transportation cost?

Solution:
Since the total destination requirement of 25 units exceeds the total resource capacity of 22 by 3 units, the problem is unbalanced.

Figure 8

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	5	8	6 [5]	6	3 [3]	8
S_2	4 [4]	7	7	6 [1]	5	5
S_3	8	4 [4]	6	6	4 [5]	9
S_{excess}	0	0	0	0 [3]	0	3
Demand	4	4	5	4	8	25

The excess requirement is handled by adding a dummy plant, s excess with a capacity equal to 3 units. We use zero unit transportation cost to the dummy plant. The modified transportation table is as follows where we have done the allocation. This initial solution is obtained by using the vogel's Approximation method and we find that there are around 7 occupied cells, the initial solution is degenerate because the problem consist of 4 rows and 5 columns and we have the feasible solution as 8 but occupied cells are 7.

In order to remove this degeneracy, we assign delta to the unoccupied cell S2, D5, which has the minimum cost amongst the unoccupied cells as shown in the table below.

Figure 9

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	5	8	6 (-) [5]	6	3 (+) [3]	8	$U_1=0$
S_2	4 [4]	7	7	6 (+) [1]	5 (-) [Δ]	5	$U_2=2$
S_3	8	4 [4]	6	6	4 [5]	9	$U_3=1$
S_{excess}	0	0	0 (+) -2	0 (-)	0	3	$U_4=-4$
Demand	4	4	5	4	8	25	
V_j	$V_1=2$	$V_2=3$	$V_3=6$	$V_4=4$	$V_5=3$		

Next we determine the ui and the vj for occupied cells as shown in the next table. Since the opportunity cost in the cell (S excess, D3) is largest negative, it must enter the basis and the cell (S2,D5) must leave the basis.

The new solution is shown as follows.

Figure 10

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	5	8	6	6	3 [8]	8
S_2	4 [4]	7	7	6 [1]	5	5
S_3	8	4 [4]	6 [2]	6 [3]	4	9
S_{excess}	0	0	0 [3]	0	0	3
Demand	4	4	5	4	8	25

Repeat the procedure of testing optimality of the solution and the optimal solution is arrived at the following table from which we calculate the minimum total transportation cost associated, that is the total cost is equal to 4 into 4 plus 4 into 4 plus 6 into 2 plus 6 into 3 plus 6 into 1 plus 3 into 8 into 10 is equal to Rs 920.

3. Degeneracy and It's Resolution - (Case 2)

Case 2: Degeneracy at subsequent iterations:
To resolve degeneracy, which occurs during optimality test, the quantity may be allocated to one or more cells that have recently been unoccupied, to have m plus n minus 1 number of occupied cells in the new solution.

Let us take an example to understand the concept better,
Goods have to transport from sources S1, S2 and S3 to destinations D1, D2 and D3. The transportation cost per unit, capacities of the sources, and the requirements of the destinations are given in the following table:

Figure 11

	D_1	D_2	D_3	SUPPLY
S_1	8	5	6	120
S_2	15	10	12	80
S_3	3	9	10	80
DEMAND	150	80	50	280

In the table, second row represents the transportation cost from source of supply s1 to destination d1, d2 & d3. Similarly third & fourth row represents the transportation cost from s2 & s3 to different destinations respectively.
Last row & column shows the demand & supply where total comes to 280.
Determine a transport schedule so that cost is minimized.

Solution:
By using the north west corner method, we get the non-degenerate initial basic feasible solution in the following table where allocation are seen in 5 occupied cells.

Figure 12

	D_1	D_2	D_3	SUPPLY
S_1	8 [120]	5	6	120
S_2	15 [30]	10 [50]	12	80
S_3	3	9 [30]	10 [50]	80
DEMAND	150	80	50	**280**

As a next step we will calculate the ui, vj and dij where in the ui, values are minus 7, 0 and minus 1, the values of vj are 15, 10 and 11. Since the unoccupied cell (S3, D1) has the largest negative opportunity cost of minus 11, therefore, cell (S3, D1) is entered into a new solution mix. The closed path for (S3, D1) is shown in the table.

Figure 13

	D_1	D_2	D_3	Supply	u_i
S_1	8 120	5 +2	6 +2	120	U_1=-7
S_2	15 (-) 30	10 (+) 50	12 +1	80	U_2=0
S_3	3 (+) -11	9 (-) 30	10 50	80	U_3=-1
Demand	150	80	50	280	
V_j	V_1=15	V_2=10	V_3=11		

The maximum allocation to (S3, D1) is 30. However, when this amount is allocated to (S3, D1) both cells (S2,D1) and (S3,D2) become unoccupied because these two have same allocations. Thus the number of positive allocations become less than the required number m plus n minus 1 is equal to 3 plus 3 minus 1 is equal to 5. Hence, this is a degenerate solution as shown in the table.

Figure 14

	D_1	D_2	D_3	SUPPLY
S_1	8 120	5	6	120
S_2	15	10 80	12	80
S_3	3 30	9 Δ	10 50	80
DEMAND	150	80	50	**280**

To remove the degeneracy, a quantity delta is assigned to one of the cells that has become unoccupied, so that there are m plus n minus 1 occupied cells. Assign delta to either (S2,D1) or (S3,D2) and proceed with the usual solution procedure. The optimal solution is given in the table.

Figure 15

	D_1	D_2	D_3	SUPPLY
S_1	8 [70]	5	6 [50]	120
S_2	15	10 [80]	12	80
S_3	3 [80]	9	10	80
DEMAND	150	80	50	**280**

4. Alternative Optimal Solution & Prohibited Transportation Routes

Alternative Optimal solution:
The existence of alternative optimal solutions can be determined by an inspection of the opportunity costs, dij for the unoccupied cells.
If an unoccupied cell in an optimal solution has an opportunity cost of zero, an alternative optimal solution can be formed with another set of allocations, without increasing the total transportation cost.

Prohibited Transportation Routes
Situations like road hazards such as snow, flood, etc, traffic regulations, etc may arise because of which it is usually not possible to transport goods from certain sources to certain destinations.
Such type of problems can be handled but by assigning a very large cost, say M or infinity to that route or cell.

Consider the problem of scheduling the weekly production of certain items for the next four weeks. The production cost of the item is Rs10 for the first two weeks and Rs.15 for the last two weeks. The weekly demands are 300, 700, 900 and 800, which must be met.

Figure 16

Production cost	**I**	**II**	**III**	**IV**
	Rs.10	Rs.10	Rs.15	Rs.15
Weekly demand	300	700	900	800

The plant can produce a maximum of 700 units per week.
In addition the company can use overtime during the second and third week. This increases the weekly production by an additional 200 units, but the production cost also increases by Rs 5.
Excess production can be stored at a unit cost of Rs 3 per week.
How should the production be scheduled so as to minimize the total cost?

Solution:
The given information is presented as a transportation problem in the table.

Figure 17

Week (origin)	Production cost per week					Supply
	I	II	III	IV	Dummy	
R_1	10	13	16	19	0	700
R_2	-	10	13	16	0	700
O_2	-	15	18	21	0	200
R_3	-	-	15	18	0	700
O_3	-	-	20	23	0	200
R_4	-	-	-	15	0	700
Demand	300	700	900	800	500	3200

The cost elements in each cell are determined by adding the production cost, the over-time cost of Rs 5 and the storage cost of Rs 3.
Thus, in the first row, the cost of Rs 3 is added during the second week onward. Since the output of any period cannot be used in a period preceding it, the cost element is written in the appropriate cells. A dummy column has been added because the supply exceeds the demand.

The problem can be solved using the MODI method for the usual transportation problem. Degeneracy occurs at the initial stage if the initial basic feasible solution is obtained by the Vogel's method. Degeneracy may be removed by adding delta in the cell R2, dummy, which provides the optimal solution.

Figure 18

	I	II	III	IV	Dummy	Supply
R_1	10 [300]	13	16 [200]	19 [100]	0 [100]	700
R_2	-	10 [700]	13 [Δ]	16	0	700
O_2	-	15	18	21	0 [200]	200
R_3	-	-	15 [700]	18	0	700
O_3	-	-	20	23	0 [200]	200
R_4	-	-	-	15 [700]	0	700
Demand	300	700	900	800	500	**3200**

We also take the production schedule in the table where the columns represent the production in week, units and the use of the units for the week. The total minimum cost for the optimal production schedule as given in the table is total cost is equal to 10 into 300 plus 16 into 200 plus 19 into 100 plus 10 into 700 plus 15 into 700 plus 15 into 700 is equal to Rs thirty six thousand one hundred.

Figure 19

Production in week	Units	For use in week I	II	III	IV
I	700	300	-	200	100
II	700	-	700	-	-
	NIL				
III	700	-	-	700	-
	NIL				
IV	700	-	-	-	700
DEMAND		300	700	900	800

5. Maximization Transportation Problem & Trans-Shipment Problem

Maximization transportation problem:
In general the transportation model is used for cost minimization problems. However, if it is also used to solve the problems in which the objective is to maximize total value or benefit. That is instead of unit cost Cij, the unit profit or payoff Pij associated with each route (i, j) is given.

The objective function in terms of total profit or payoff is then stated as follows:
maximize Z is equal to summation of Pij into Xij.
The algorithm for solving this problem is same as that for the minimization problem. However, since we are given profits instead of costs, therefore a few adjustments in vogels approximation method for finding the initial solution and in the MODI optimality test are required.

For finding the initial solution by VAM, the penalties are computed as the difference between the largest and next largest payoff in each row or column.
In this case, row and column differences represent payoffs.
Allocations are made in those cells where the payoff's is largest, corresponding to the highest row or column differences.

Since it is maximization problem, the criterion of optimality is the converse of the rule for minimization.
The rule is : A solution is optimal if all opportunity costs for dij for the unoccupied cells are zero or negative.

Trans-shipment problem:
In a transportation problem the shipment of a commodity takes place among sources and destinations.
But instead of direct shipments to destinations, the commodity can be transported to a particular destinations through one or more intermediate or trans-shipment points.
Each of these points, in turn supply to other points.

Thus, when the shipments pass from destination to destination and from source to source, we have a trans-shipment problem.
The solution to this problem can be obtained by using the transportation model.
The solution procedure is as follows: if there are m sources and n destinations, we shall have a transportation table of size m plus n into m plus n instead of m into n as in the usual case.

If the total number of units transported from all sources to all destinations is N, then the given supply at each source and demand at each destination are set to be equal to N. The problem can then be solved by the usual MODI method for transportation problems.

In the final solution, ignore the units transported from a point to itself, that is diagonal cells, because they do not have any physical meaning (no transportation).

Here's a summary of our learning in this session, where we have understood:

- The variation in transportation problem
- The maximization in transportation problem
- The trans-shipment problem