1. Introduction

Welcome to the series of E-learning modules on Transportation problem & Initial Solution by the North West Corner Rule.

By the end of this session, you will be able to:

- Recognize and formulate a transportation problem involving a large number of shipping routes
- Drive initial feasible solution using the North West Corner Method

Let us start with an Introduction:
A programming problem that is concerned with the optimal pattern of the distribution of goods from several points of origin to several different destinations, with the specified requirements at each destination is called as transportation problem.

One important application of linear programming has been in the area of the physical distribution or transportation of resources, from one place to another, to meet a specific set of requirements.
It is easy to express a transportation problem mathematically in terms of an LP model, which can be solved by the simplex method.
Since a transportation problem involves a large number of variables and constraints, it takes a very long time to solve it by the simplex method.
Simpler ways, involving transportation algorithms have been evolved for this purpose, namely Stepping Stone Method and the MODI (modified distribution) Method.

The structure of transportation problem involves a large number of shipping routes from several supply origins to several demand destinations.
The objective is to determine the number of units of an item that should be shipped from an origin to a destination to satisfy the required quantity of goods or services at each destination center.
This should be done within the limited quantity of goods or services available at each supply center, at the minimum transportation cost and or time.

The transportation algorithm discussed is applied to minimize the total cost of transporting a homogeneous commodity from one supply center to demand centers.
However, it can also be applied to the maximization of some total value or utility; for example, financial resources are distributed in such a way that the profitable return is maximized.
There are various types of transportation models and the simplest of them was first presented by F L Hitchcock in the year 1941 and later on further development was contributed by T C Koopmans in the year 1949 and G B Dantzig in the year 1951.
Several extensions of transportation models and methods have been subsequently developed.

2. Mathematical Model of Transportation Problem

Let us consider an example to illustrate the mathematical model formulation of transportation problem of transporting a single commodity from three sources of supply to four demand destinations.

The sources of supply are production facilities, warehouses or supply points, characterized by available capacities.
The destinations are consumption facilities, warehouses or demand points, characterized by required levels of demand.

Let us illustrate with an example:

Figure 1

Production facility	Production capacity/ week (in 100's)	Warehouses	Demand/ week (in 100's)
S_1	7	D_1	5
S_2	9	D_2	8
S_3	18	D_3	7
		D_4	14

A company has three production facilities S1, S2 and S3 with production capacity of 7, 9 and 18 units in 100's per week of a product respectively. These units are to be shipped to four warehouses D1, D2, D3 and D4 with requirement of 5, 8, 7 and 14 (in 100's) per week, respectively.

The transportation costs (in Rupees) per unit between factories to warehouses are given in the table.

Figure 2

From \ To	D_1	D_2	D_3	D_4	Capacity
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	**34**

In the table, we have taken the sources of supply in the rows and the units of demand in the columns. The values of cost from the source to the destination is indicated
Second row shows the transportation cost from production facility s1 to warehouse d1which is 19, to d2 is 30, to d3 is 50 & to d4 is 10.
Similarly, third & fourth row shows the transportation cost from production facility s2 & s3 to warehouse d2, d3 & d4 respectively.
Last row & last column indicates the demand & production capacity respectively where total comes to 34.
Formulate this transportation problem as a Linear Programming model to minimize the total transportation cost.

Model formulation:
Let 'Xij' is equal to number of units of the product to be transported from factory 'i' (where, 'i' is equal to 1, 2, 3) to ware house 'j' (where, 'j' is equal to 1, 2, 3, 4).
The transportation problem is stated as an linear programming model as follows:
Minimize (total transportation cost) Z is equal to 19'X'11 plus 30'X'12 plus 50'X'13 plus 10'X'14 plus 70'X'21 plus 30'X'22 plus 40'X'23 plus 60'X'24 plus 40'X'31 plus 8'X'32 plus 70'X'33 plus 20'X'34.

Subject to the constraints,
X11 plus X12 plus X13 plus X14 is equal to 7, X21 plus X22 plus X23 plus X24 is equal to 9 and X31 plus X32 plus X33 plus X34 is equal to 18 are the capacity available.

X11 plus X21 plus X31 is equal to 5, X12 plus X22 plus X32 is equal to 8, X13 plus X23 plus X33 is equal to 7 and X14 plus X24 plus X34 is equal to 14 are the demand, where, 'Xij' is greater than equal to zero for I and j.

In the above linear programming model, there is m by n matrix that is a 3 by 4 matrix with 12 decision variables, Xij and m plus n with 7 constraints, where 'm' is the number of rows and 'n' is the number of columns in a general transportation table.

The transportation problem applies to situations in which a single product is to be transported from several sources, also called origin or supply or capacity centers, to several sinks, also called destination or demand or requirements centers.

In general, let there will be 'm' sources S1, S2 up to Sm having ai (where I equal to 1,2 up to 'm') units of supplies or capacity respectively to be transported among 'n' destinations, D1, D2 up to Dn with b_j (where 'j' equal to 1,2 up to n) units of requirements respectively.

Let 'cij' be the cost of shipping one unit of the commodity from source 'i' to destination 'j' for each route.
If 'xij' represents the units shipped per route from source 'i' to destination 'j', then the problem is to determine the transportation schedule so as to minimize the total transportation cost satisfying supply and demand conditions.
Mathematically, the problem may be stated as follows:

Minimize (total cost) Z = Summation ‘cij’ ‘xij’ (where, limits i=1, j=1).
Subject to constrains,
Summation ‘xij’ equal to ‘ai’, where, i=1,2 up to m (that is, supply constraints)
Summation ‘xij’ equal to ‘bi’, where, j=1,2 up to n (that is, supply constraints)
‘xij’ greater than or equal to zero for all ‘i’ and ‘j’.

For ease in presentation and solution, a transportation problem is generally presented as shown in table:

Figure 3

To / from	D_1	D_2	.....	D_n	Supply (a_i)
S_1	C_{11} x_{11}	C_{12} x_{12}	.....	C_{1n} x_{1n}	a_1
S_2	C_{21} x_{21}	C_{22} x_{22}	.....	C_{2n} x_{2n}	a_2
:	:	:	:	:	:
S_m	C_{m1} x_{m1}	C_{m2} x_{m2}	......	C_{mn} x_{mn}	a_m
Demand (b_j)	B_1	B_2	......	B_n	$\Sigma a_i = \Sigma b_j$

In the table, we have taken the sources of supply in the rows and the units of demand in the columns. The values of cost from the source to the destination is indicated in the top left corner of the box as ‘Cij’ and the number of units from source to destination is indicated in the center of the box ‘Xij’.
The grand total of the rows and columns are always equal and indicated as summation ‘ai’ and summation ‘bi’, the total of the rows are indicated as ‘ai’ and total of the columns are indicated as ‘bi’.

Existence of feasible solution:
A necessary and sufficient condition for the existence of a feasible solution to the transportation problem is Total supply is equal to Total demand.
Summation ‘ai’ is equal to Summation ‘bj’ (where, limits i=1 and j=1) (also called rim condition).that is, the total capacity or supply must equal the total requirement or demand.

Remarks:

1. When the total supply equals total demand, the problem is called a balanced transportation problem, otherwise an unbalanced transportation problem

2. When the number of positive allocations (that is, values of decision variables) at any stage of the feasible solution is less than the required number (rows plus columns minus one), that is number of independent constraint equations, the solution is said to be degenerate, otherwise non-degenerate

3. Cells in the transportation table having a positive allocation will be called occupied cells, otherwise empty or non-occupied cells

3. Loops in Transportation Table & Their Properties

Loops in transportation table and their properties are discussed below:
In this problem, there are 'm' plus 'n' constraints, one for each source of supply and distinction and 'm' by 'n' variables.
Since all 'm' plus 'n' constraints are equations, and since the transportation model is always balanced that is, total supply is equal to total demand, one of these equations is extra (that is, redundant).

The extra constraint equation can be derived from other constraint equations, without affecting the feasible solution.
It follows that any feasible solution for a transportation problem must have exactly 'm' plus 'n' minus non negative basic variables or allocations 'Xij' satisfying the rim conditions.

An ordered set of at least four cells in a transportation table is said to form a loop provided:

- Any two adjacent cells of the ordered set lie either in the same row or in the same column, and
- No three or more adjacent cells in the ordered set lie in the same row or column.

The first cell of the set must follow the last in the set, that is, each cell except the last cell must appear only once in the ordered set.

Remarks:

1. Every loop has an even number of cells and at least four
2. The allocations are said to be in independent position if it is not possible to increase or decrease any independent individual allocation without changing the positions of these allocations, or violating the rim conditions, a closed loop cannot be formed through these allocations
3. Each row and column in the transportation table should have only one plus and minus sign, all cells that have a plus or minus sign, except the starting unoccupied cell, must be occupied cells
4. Closed loops may or may not be square in shape

4. The Transportation Method

The solution algorithm to a transportation problem may be summarized into the following steps:

Step 1: Formulate the problem and set up in the matrix form.

- The formulation of the transportation problem is similar to the linear programming problem formulation
- Here the objective function is the total transportation cost and the constraints are the supply and demand available at each source and destination respectively

Step 2: Obtain an initial basic feasible solution.

In practice there are three different methods to obtain an initial solution, that is,

i. North-West corner Method
ii. Least Cost Method, and
iii. Vogel's Approximation or penalty Method

The initial solution obtained by any of the three methods must satisfy the following conditions:

a) The solution must be feasible, that is, it must satisfy all the supply and demand constraints (also called rim conditions).

b) The number of positive allocations must be equal to 'm' plus 'n' minus one, when 'm' is the number of rows and 'n' is the number of columns.

Any solution that satisfies the above conditions is called non-degenerate basic feasible solutions, otherwise, degenerate solution.

Step 3: Test the initial solution for optimality.

We shall discuss the Modified Distribution (MODI) method to test the optimality of the solution.

If the current solution is optimal, then stop. Otherwise, determine the new improved solution.

Step 4: Updating the solution.

Repeat step 3 until an optimal solution is reached.

There are several methods available to obtain an initial solution. Here we shall discuss the North west Corner Method:

North-West Corner Method (NWCM) – It is a simple and an efficient method to obtain an initial solution.

This method does not take into account the cost of transportation on any route of transportation.

The method can be summarized as follows:

Step 1: Start with the cell at the upper left (north-west) corner of the transportation matrix and allocate commodity equal to the minimum of the rim values for the first row and first column, that is, minimum (a1, b1).

Step 2:

1. If allocation made in step 1 is equal to the supply available at first source (that is, a1, in first row), then move vertically down to the cell (2, 1) in the second row and the first column and apply step 1 again, for next allocation

2. If allocation made in step 1 is equal to the requirement of the first destination (that is, b1 in first column), then move horizontally to the cell (1, 2) in the first row and second column and apply step 1 again for next allocation

3. If a1 equal to b1, allocate x11 equal to a, or b1 and move diagonally to the cell (2, 2)

Step 3: Continue the procedure step by step till an allocation is made in the south-east corner cell of the transportation table

Remarks:

If during the process of making allocations for a particular cell, the supply equals demand, then the next allocation of magnitude zero can be made in cell either in the next row or column.

This condition is known as degeneracy and may also occur in later steps.

5. Illustrations

Let us look at the Illustration-1:

Figure 4

Production facility	Production capacity/ week (in 100's)	Warehouses	Demand/ week (in 100's)
S_1	7	D_1	5
S_2	9	D_2	8
S_3	18	D_3	7
		D_4	14

A company has three production facilities S1, S2 and S3 with production capacity of 7, 9 and 18 units (in 100's) per week of a product respectively. These units are to be shipped to four warehouses D1, D2, D3 and D4 with requirement of 5, 8, 7 and 14 (in 100's) per week, respectively.

The transportation costs (in Rupees) per unit between factories to warehouses are given in the table.

Figure 5

From \ To	D_1	D_2	D_3	D_4	Capacity
s_1	19	30	50	10	7
s_2	70	30	40	60	9
s_3	40	8	70	20	18
Demand	5	8	7	14	**34**

In the table, we have taken the sources of supply in the rows and the units of demand in the columns. The values of cost from the source to the destination is indicated
Second row shows the transportation cost from production facility s1 to warehouse d1which is 19, to d2 is 30, to d3 is 50 & to d4 is 10.
Similarly, third & fourth row shows the transportation cost from production facility s2 & s3 to warehouse d2, d3 & d4 respectively.
Last row & last column indicates the demand & production capacity respectively where total comes to 34.

Use North West Corner Method to find an initial basic feasible solution to the transportation problem.

Solution:

Figure 6

To / From	D_1	D_2	D_3	D_4	Capacity
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	34

The cell (S1, D1) is the North West corner cell in the given transportation table. The rim values for row S1 and column D1 are compared. The smaller of the two, that is five, is assigned as the first allocation; otherwise it will violate the feasible condition. This means that five units of a commodity are to be transported from source S1 to destination D1. However, this allocation leaves a supply of seven minus five that is equal to two units of commodity at S1.

Figure 7

To / From	D_1	D_2	D_3	D_4	Capacity
S_1	19 $\boxed{5}$	30 $\boxed{2}$	50	10	7
S_2	70	30 $\boxed{6}$	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	34

Move horizontally and allocate as much as possible to cell (S1, D2). The rim value for row S1 is two and for column D2 is eight. The smaller of the two, that is two, is placed in the cell.

Proceeding to row S2, since demand requirement of D1 has been met; nothing further can be allocated to D1. The unfulfilled requirement of D_2 is now eight minus two that is equal to six units. This can be fulfilled by S2 with capacity of nine units. So six units are allocated to cell (S2, D2). The requirement of D2 is now satisfied and a balance of nine minus six that is equal to three units remains with S2.

Figure 8

To / From	D_1	D_2	D_3	D_4	Capacity
S_1	19 $\boxed{5}$	30 $\boxed{2}$	50	10	7
S_2	70	30 $\boxed{6}$	40 $\boxed{3}$	60	9
S_3	40	8	70 $\boxed{4}$	20 $\boxed{14}$	18
Demand	5	8	7	14	34

By moving in the same manner horizontally and vertically as successive requirement and capacity are met, we ensure that the solution is feasible, that is, number of positive allocations (occupied cells) is equal to 'm' plus 'n' minus one equal to three plus four minus one that is equal to six.

The total transportation cost of the initial solution derived by the North West corner method is obtained by multiplying the quantity 'Xij' in the occupied cells with the corresponding unit cost 'Cij' and adding all the values together. Thus the total transportation cost of this solution is five into nineteen plus two into thirty plus six into thirty plus three into forty plus four into seventy plus fourteen into twenty, which is equal to one thousand fifteen rupees.

Illustration 2:
A diary firm has three plants located in a state. The daily milk production at each plant is as follows:

Figure 9

Diary plant	Daily milk production (in million liters)	Distribution Center	Requirements of distribution Center (in million liters)
P_1	6	D_1	7
P_2	1	D_2	5
P_3	10	D_3	3
		D_4	2

Plant 1- 6 million liters, Plant 2- 1 million liters and Plant 3 – 10 million liters. Each day, the firm must fulfill the needs of its four distribution centers.
The minimum requirement of each center is as follows: distribution center 1- 7 million liters, distribution center 2- 5 million liters, distribution center 3- 3 million liters, distribution center 4-2 million liters.

Cost of shipping of one million liter from each plant to each distribution center is given in the following table:

Figure 10

Plant	Distribution Center			
	D_1	D_2	D_3	D_4
P_1	2	3	11	7
P_2	1	0	6	1
P_3	5	8	15	9

Cost of shipping from p1 to D1, D2, D3 & D4 are 2,3,11 & 7 respectively.
Similarly, cost of shipping from plant p2 & p3 to various distribution centers respectively are given below.
Find the initial basic feasible solution for the given problem using North West Corner Method.

Solution:

Figure 11

Plant	Distribution Center				
	D_1	D_2	D_3	D_4	Supply
P_1	2 $\boxed{6}$	3	11	7	6=a1
P_2	1 $\boxed{1}$	0	6	1	1=a2
P_3	5	8 $\boxed{5}$	15 $\boxed{3}$	9 $\boxed{2}$	10=a3
Demand	7=b1	5=b2	3=b3	2=b4	

In the following table, we have the columns showing the distribution center D1, D2, D3 and D4.
We have represented the rows of the table with the plants producing milk P1, P2 and P3 so we get the supply as a1 equal to 6, a2 equal to 1 and a3 is equal to 10, similarly we have the demand b1 equal to 7 b2 equal to 5, b3 is equal to 3 and b4 is equal to 2.
Step one includes comparing a1 and b1, since a1 is less than b1, allocate x11 is equal to 6. This exhausts the supply of P1 and leaves 1 unit as unsatisfied demand at D1.
Move to cell (P2, D1). Compare a2 and b1 (that is 1 and 1). Since a2 is equal to b1, allocate x21 is equal to 1.
Move to cell (P3, D2), since supply at P3, is equal to the demand at D2, D3 and D4, therefore allocate x32 is equal to 5, x33 is equal to 3 and x34 is equal to 2.

It may be noted that the number of allocated cells 9 also called basic cells are 5 which is one less than the required number 'm' plus 'n' minus 1 (that is, 3 plus 4 minus 1) is equal to 6.
Thus, the solution is the degenerate solution, transportation cost associated with this solution is total cost is equal to Rs. (6 into 2 plus 1 into 1 plus 5 into 8 plus 3 into 15 plus 2 into 9) into 100 is equal to Rs. 11 thousand six hundred.

Here's a summary of our learning in this session, where we have understood:

- To Recognize and formulate a transportation problem involving a large number of shipping routes
- To Drive initial feasible solution using the North West Corner Method