Reliability Measures

Part of

Paper XIII: Statistical Quality Control (SQC) and Reliability

(Prerequisite: Paper V : Probability distributions -2)

Introduction:

We depend on working of a systems such as machines, electronic components, or even business systems. Any system involves one or more components. These components are likely to fail to perform the required functions after working for some time. For example, a computer consists of many electronic components. If one of the components fails, the computer may not work. In some of the systems even if one of the components fail, still the system may work but its reliability is reduced. When we study the quality of a component, its status of working condition is verified in the beginning, whereas many of the units which are perfectly working in the beginning will not meet the specification after a month, or after a year or after the warranty period. Hence, quality is decided at the start and reliability is motion picture of the day by day operation. Here we identify the condition through certain measures and that will reflect the reliability of the system. This measurement will help a business to assess the possible loss may be incur. The reliability measure of the system will depend on reliability measures of components. We study these interrelations as well.

Reliability

The theory of reliability is the analysis of failures of systems that are operating, their causes and within predetermined limits under given operating conditions. Any complex systems, even if properly designed, they are prone to failure. Study of failure properties of a component of a system is the most important characteristic of product quality as things have to be working satisfactorily before considering other quality attributes. Reliability is an internal property of operating systems and, as such, it has to be specified as part of the systems' characteristics. Hence it is necessary to develop quantitative basis for describing the failure process of systems to measure their reliability as accurately as possible. This enables one to compare the performance of similar systems on a reliability basis and to develop methods for improving the reliability of a particular system. Here we discuss the statistical basis of the theory of reliability and assume the reader is familiar with various probability distribution functions of random variables.

Usually, specific performance measures can be embedded into reliability analysis by the fact that if the performance is below a certain level, a failure can be said to have occurred. The term reliability is used generally to express a certain degree of assurance that a device or a system will operate successfully in a specified environment during a certain time period. The concept is a dynamic one, and does not refer just to an instantaneous event. If a device fails, this does not necessarily imply that it is unreliable. Every piece of mechanical or electronic equipment fails once in a while. The question is how frequently failures occur in specified time periods. There is often some confusion among practitioners between quality level and reliability. It is the proportion of defective computer chips manufactured daily is too high, we could say that the manufacturing process is not sufficiently reliable. The defective computer chips are scrapped yet the non-defective ones may be very reliable. The reliability of the computer chips depends on their technological design and their function in the computer. If the installed chips do not fail frequently, they might be considered reliable.

Reliability is probability of system successfully working until time t, that is, the system did not fail until the time t.

Life distributions:

The length of life of a component, which is a random variable, is the basis of reliability functions and other related functions. The length of life (lifetime) of a component/ system is the length of the time interval, T, from the initial activation of the unit until its failure. This variable, T, is considered a random variable, since the length of life cannot be exactly predicted. The probability distribution function of T, or cumulative (life) distribution function (CDF) of T, denoted by $F(t)$, is the probability that the lifetime does not exceed t, i.e., $F(t) = P\{T \leq t\}, 0 < t < \infty$.

The lifetime random variable T is called continuous if its CDF is a continuous function of t. For such life time random variables the probability density function (PDF) corresponding to $F(t)$ is defined by its derivative (of $F(t)$, if it exists). We denote the PDF by $f(t)$. This is a non-negative valued function such that

$$F(t) = \int_{-\infty}^{t} f(x)\,dx \quad \text{for all} \quad t \quad \text{from } -\infty \text{ to } \infty.$$

Since the life distributions are nonnegative valued random variables we can write

$$F(t) = \int_{0}^{t} f(x)\,dx \quad \text{for } 0 \le t < \infty$$

Reliability function:

Definition *Reliability* is the probability that the system will perform its intended function under specified working condition for a specified period of time. Mathematically, the reliability function $R(t)$ is the probability that a system will be successfully operating without failure in the interval from time 0 to time t,

$R(t) = P(T > t)$, $t \ge 0$

where T is a random variable representing the failure time or time-to-failure.

Since $P(T > t) = 1 - P\{T \le t\}$ for all t, we have following relationship between reliability function probability distribution function.

$$R(t)=1-F(t) \quad \text{for all } t \ge 0.$$

The reliability function can be written in terms of PDF when life distribution is continuous.

$$R(t) = \int_{t}^{\infty} f(x)\,dx \quad \text{for } 0 \le t < \infty$$

$$\text{Also note that} \quad \frac{d\,R(t)}{dt} = -f(t) \quad \text{for} \quad 0 \le t < \infty.$$

Hazard function

The failure rate function, or hazard function, is very important in reliability analysis because it specifies the rate of the system aging. The definition of failure rate function is given here.

Definition. The *hazard function or hazard rate or failure rate function* $\lambda(t)$ of a system is defined that $\lambda(t)\,dt$ is the conditional probability of time to failure of the system in the time interval $(t, t+dt]$, given that the system has not failed from time origin up to time t, $t \ge 0$. Thus

$$\lambda(t)\; dt = \frac{P(t < T < t + \Delta t, T > t)}{P(T > t)}$$

The numerator in the right hand expression is the joint probability that the system has survived to time t and will fail in the time interval (t, $t+dt$]; this is equivalent to the probability that the system will fail in the time interval (t, $t+dt$], since this event presumes survival of the system to time t.

Hence, $$\lambda(t)\; dt = \frac{P(t < T < t + \Delta t)}{P(T > t)} = \frac{f(t)\; dt}{R(t)}$$

$$\lambda(t) \;= \lim_{dt \to 0} \frac{P(t < T < t + \Delta t)}{dt \;\; P(T > t)} = \frac{f(t)}{R(t)}$$

Also $$\lambda(t) \;= \lim_{dt \to 0} \frac{R(t) - R(t + \Delta t)}{dt \;\; R(t)} = \lim_{dt \to 0} \frac{-[R(t + \Delta t) - R(t)]}{dt} \;\; \frac{1}{R(t)} = \frac{f(t)}{R(t)}$$

Aternatively, $$\lambda(t) \;=\; \frac{f(t)}{1 - F(t)} \quad \text{for all } t \geq 0.$$

Suppose that we have a large number of identical components which start operating at exactly the same time under identical conditions; then $\lambda(t)\; dt$ denotes the proportion of components that have survived up to time t, but that will fail in the interval (t, $t+dt$].

Now we establish the relations between hazard function and distribution function.

$$\int_0^t \lambda(x)\,dx = \int_0^t \frac{f(x)}{1-F(x)}\,dx = \int_0^t \frac{dF(x)}{1-F(x)}$$

$$= -\int_0^t \frac{d[1-F(x)]}{1-F(x)}$$

$$= -\int_0^t d\left(\log[1-F(x)]\right)$$

$$= \log[1-F(0)] - \log[1-F(t)]$$

$$= -\log\frac{1-F(t)}{1-F(0)}$$

With F(0) = 0, $\displaystyle\int_0^t \lambda(x)\,dx = -\log[1-F(t)]$

Or $\displaystyle 1-F(t) = \exp\left(-\int_0^t \lambda(x)\,dx\right)$

$$F(t) = 1-\exp\left(-\int_0^t \lambda(x)\,dx\right)$$

Following expressions for $F(t)$, $R(t)$, and $f(t)$ are given in terms of hazard function with an assumption that $F(0) = 0$,

$$F(t) = 1-\exp\left(-\int_0^t \lambda(x)\,dx\right) \quad \text{for all } t \geq 0,$$

$$R(t) = \exp\left(-\int_0^t \lambda(x)\,dx\right) \quad \text{for all } t \geq 0,$$

and $$f(t) = \lambda(t)\exp\left(-\int_0^t \lambda(x)\,dx\right) \quad \text{for all } t \geq 0.$$

Common Life Distributions:

(i) **Exponential Distribution**

The simplest form of the hazard function is a constant function, that is ,

$\lambda(t) = \lambda$ (a constant) for all t ≥ 0.

Then, $f(t) = \lambda \exp\left(-\int_0^t \lambda \, dx \right) = \lambda \exp(-\lambda t)$ for all $t > 0$.

This is the exponential density function. That is life time is exponential distribution.

The mean of this life time distribution is $1/\lambda$.

Example: A particular component is a complex mechanical system has an exponential time to failure or life time distribution with mean time to failure of 1000 hours.

a) The density function or PDF of time to failure is $f(t) = \lambda e^{-\lambda t}$, $\lambda > 0$, $t > 0$, where λ=1/1000 =0.001.
b) The reliability function $R(t)$ = exp $(-\lambda t)$ = exp(-0.001 t), $t \geq 0$.
c) If the component has to operate for 2000 hours, its reliability, which is its probability of survival after 2000 hours of operation, is Rx(2000) = exp(-0.001 x 2000) = exp(-2) =0.1352.

In case of exponential life distribution it is true that if a unit is functioning at time t and the probability of survival over an additional time period t_a is the same regardless of t, that is,

$P(T > t+ t_a \mid T > t) = P(T > t+ t_a , T > t) / P(T > t) = P(T > t+ t_a) / P(T > t)$

$= R(t+ t_a) / R(t) = exp(-\lambda (t+ t_a)) / exp(-\lambda t) = exp(-\lambda t_a) = R (t_a)$.

Weibull distribution

Frequently used to model fatigue failure, ball bearing failure etc. (very long tails)

The density function or PDF is given by

$f(t) = \lambda\alpha t^{\alpha-1} \exp(-\lambda t^{\alpha})$ for $t > 0$.

$F(t) = 1 - \exp(-\lambda t^{\alpha}) \quad \text{for } t > 0.$

The hazard rate function is $h(t) = \lambda\alpha t^{\alpha-1} \quad \text{for } t > 0.$

The reliability function is $R(t) = \exp(-\lambda t^{\alpha}) \quad \text{for } t > 0.$

This is two parameter Weibull distributions, α is called the shape parameter and λ is the scale parameter. This reduces to exponential distribution when $\alpha = 1$. We note that for the Weibull distribution the hazard function is

decreasing if $0 < \alpha < 1$

constant if $\alpha = 1$

increasing if $1 < \alpha < \infty$.

Hence, Weibull distribution is capable of modeling decreasing failure rate (DFR) ($\alpha < 1$), constant failure rate (CFR) ($\alpha = 1$) and increasing failure rate (IFR) ($\alpha > 1$) behavior.

This class of hazard function provides a practical means for fitting the various portions of the life characteristic curve of system. The initial-failure portion would be represented by hazard function with $\alpha < 1$, the random failure portion would be represented to hazard function with $\alpha = 1$, and the wear-out failure portion would be represented by a hazard function with $\alpha > 1$.

Discrete distributions

Discrete distributions are also used to model distributions in reliability. The binomial distribution is one of the most widely used discrete random variable distributions in reliability and quality inspection. It has applications in reliability engineering, *e.g.,* when one is dealing with a situation in which an event is either a success or a failure.

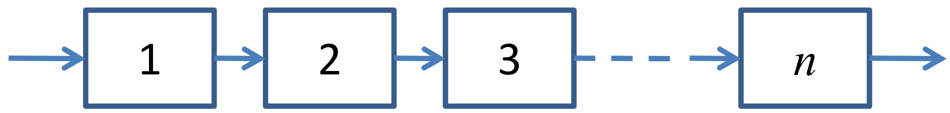
Although the Poisson distribution can be used in a manner similar to the binomial distribution, it is used to deal with events in which the sample size is unknown.

SERIES AND PARALLEL SYSTEMS

Reliability of a Series System

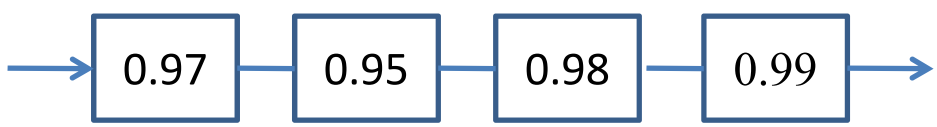
Many physical and non-physical systems (e.g. air-conditioning systems, biological and ecological systems, quality control systems in manufacturing plants, etc.) may be viewed as assemblies of many interacting elements. The elements are often arranged in mechanical or logical series or parallel configurations.

A series system works only when all their components work. Examples



The probability that a series system works is same as the probability that simultaneously all of its components are working. Hence, the probability of the system working is same as the product of probabilities of individual components working.

A system consists of 4 components in series each having a reliability as indicated here. What is the reliability of the system?



Reliability of the system is $= 0.97 \times 0.95 \times 0.98 \times 0.99 = 0.894093$

A system consists of 8 components in series each having a reliability of 0.98 then the reliability of the system is $(0.98)^8 = 0.850763$.

An electronic product contains 100 integrated circuits. The probability that any integrated circuit is defective is .001 and the integrated circuits are independent. The product operates only if all the integrated circuits are operational. What is the probability that the product is operational?

Solution:

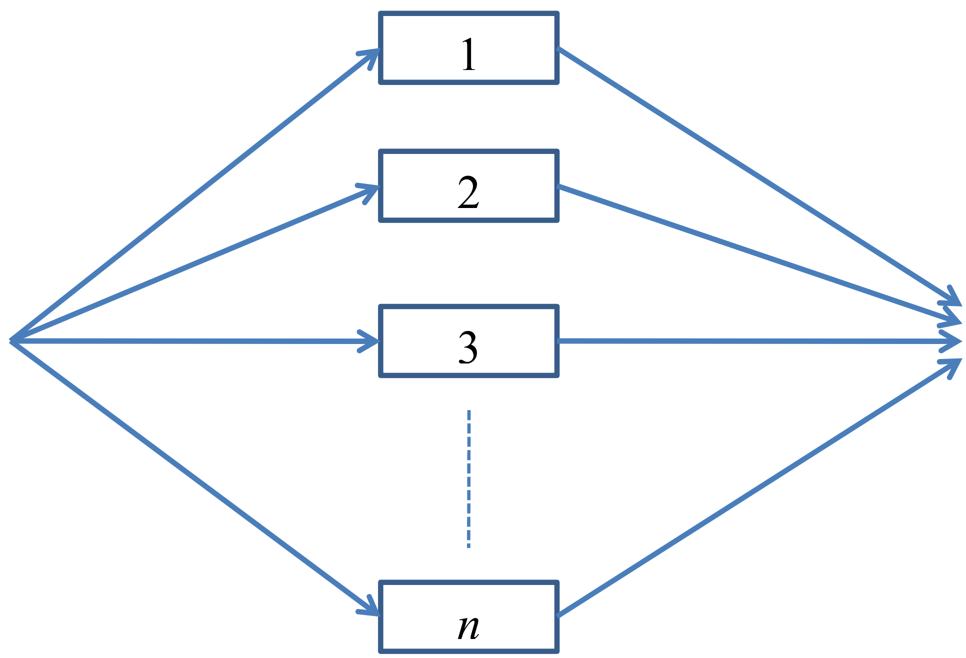
The probability that any component is functioning is .999. Since the product operates only if all 100 components are operational, the probability that the 100 components are functioning is: $(0.999)^{100} = 0.904792$.

The reliability is just over 90% even though each component has a reliability of 99.9%.

Reliability of a Parallel System

Parallel system fails only if all its components fail. For example, if an office has n copy machines, it is possible to copy a document if at least one machine is in good working conditions. The system fails only when all components fail. Hence, the probability of failure of a parallel system is the product of the probability of the failure of all the individual components. If $P_1, P_2, \ldots\ldots P_n$ are the reliability of individual components then the reliability of the system is

$$1-(1-P_1)*(1-P_2)*(1-P_3)*\ldots\ldots\ldots*(1-P_n).$$



Examples

A system consists of 4 components in parallel. If reliability of these components are 0.9, 0.85, 0.8, 0.9 respectively, then the overall reliability of the system is = 1- 0.1 x 0.15 x 0.2 x 0.1 = 0.9997. That is 99.97%

A system has three components and probabilities of failure of these components are 0.05, 0.1 and 0.15 respectively. What is the probability that the system is reliable?

The system reliability is 1 - 0.05 x 0.1 x 0.15 = 0.99925.

Practicals:

For all positive t, following expressions hold.

	Exponential Distribution	Weibull Distribution
Density function	$\lambda\, exp(-\lambda t)$	$\lambda \alpha\, t^{\alpha-1} \exp(-\lambda\, t^{\alpha})$
Distribution function	1- $exp(-\lambda t)$	$1-\exp(-\lambda\, t^{\alpha})$
Reliability function	$exp(-\lambda t)$	$\exp(-\lambda\, t^{\alpha})$
Hazard function	Λ	$\lambda \alpha\, t^{\alpha-1}$

Let X be have exponential distribution then,

Parameter λ = 0.5

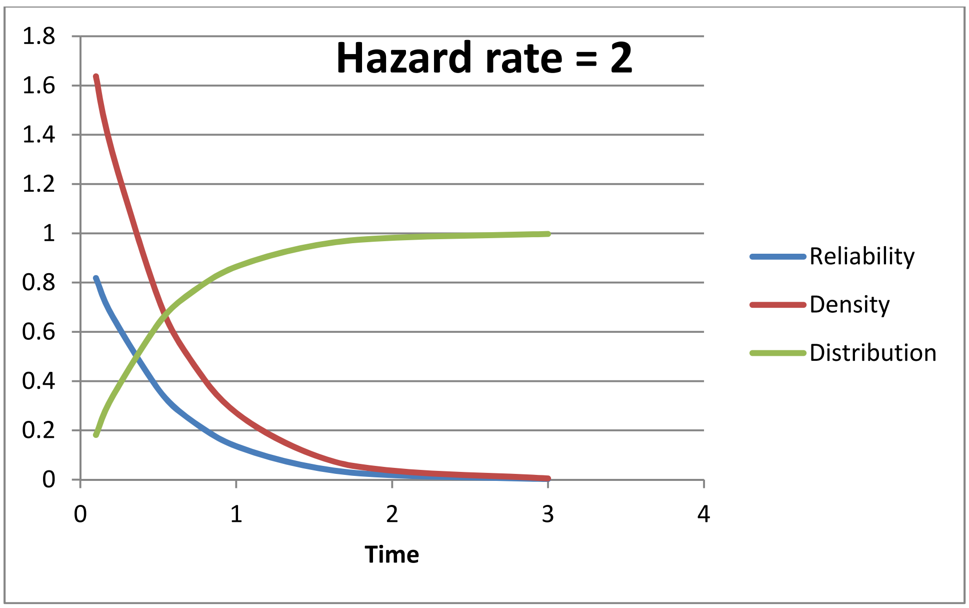
Time	t	0.1	0.2	0.5	0.7	1	1.5	2	3	5
Hazard function	λ	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
Reliability function	$exp(-\lambda t)$	0.9512	0.9048	0.7788	0.7047	0.6065	0.4724	0.3679	0.2231	0.0821
Density function	$\lambda\, exp(-\lambda t)$	0.4756	0.4524	0.3894	0.3523	0.3033	0.2362	0.1839	0.1116	0.0410
Distribution function	1- $exp(-\lambda t)$	0.0488	0.0952	0.2212	0.2953	0.3935	0.5276	0.6321	0.7769	0.9179

Parameter λ = 1.0

Time	t	0.1	0.2	0.5	0.7	1	1.5	2	3	5
Hazard function	λ	1	1	1	1	1	1	1	1	1
Reliability function	$exp(-\lambda t)$	0.9048	0.8187	0.6065	0.4966	0.3679	0.2231	0.1353	0.0498	0.0067
Density function	$\lambda\, exp(-\lambda t)$	0.9048	0.8187	0.6065	0.4966	0.3679	0.2231	0.1353	0.0498	0.0067
Distribution function	1- $exp(-\lambda t)$	0.0952	0.1813	0.3935	0.5034	0.6321	0.7769	0.8647	0.9502	0.9933

Parameter λ = 2.0

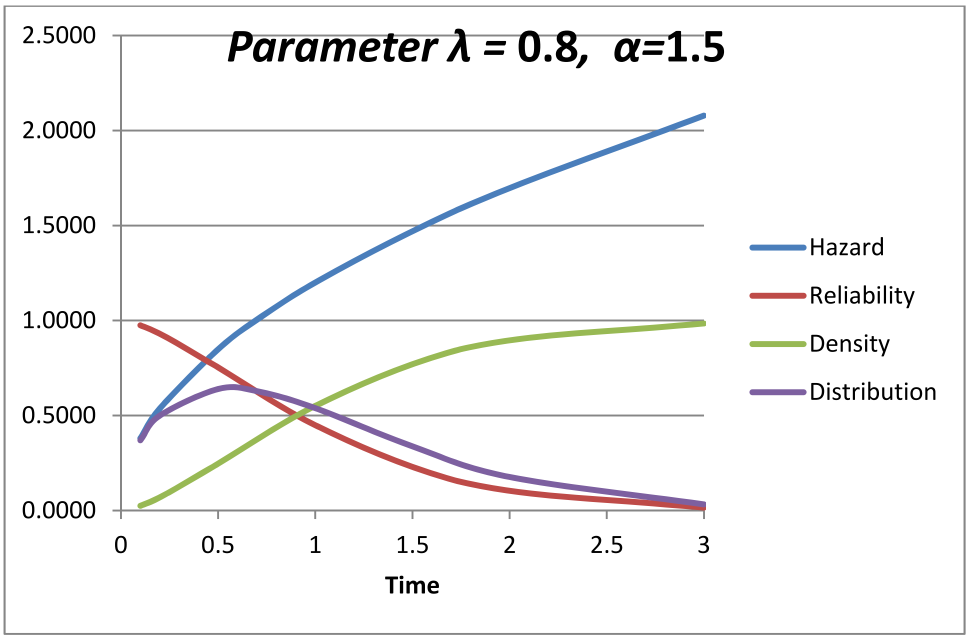
Time	t	0.1	0.2	0.5	0.7	1	1.5	2	3	5
Hazard function	λ	2	2	2	2	2	2	2	2	2
Reliability function	$exp(-\lambda t)$	0.8187	0.6703	0.3679	0.2466	0.1353	0.0498	0.0183	0.0025	0.0000
Density function	$\lambda\ exp(-\lambda t)$	1.6375	1.3406	0.7358	0.4932	0.2707	0.0996	0.0366	0.0050	0.0001
Distribution function	$1\text{-}\ exp(-\lambda t)$	0.1813	0.3297	0.6321	0.7534	0.8647	0.9502	0.9817	0.9975	1.0000



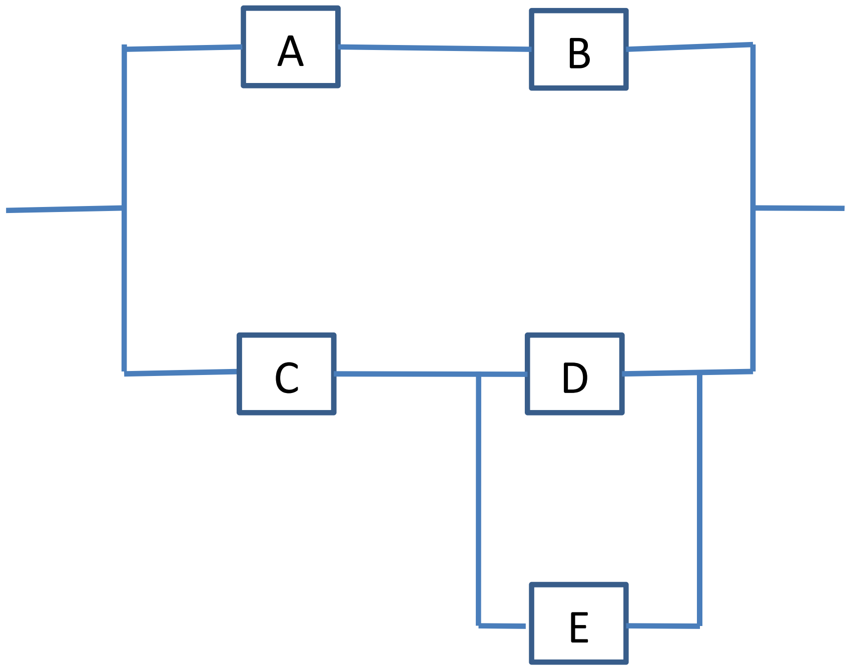
Weibull distribution

Parameter $\lambda = 0.8,\ \ \alpha$=1.5

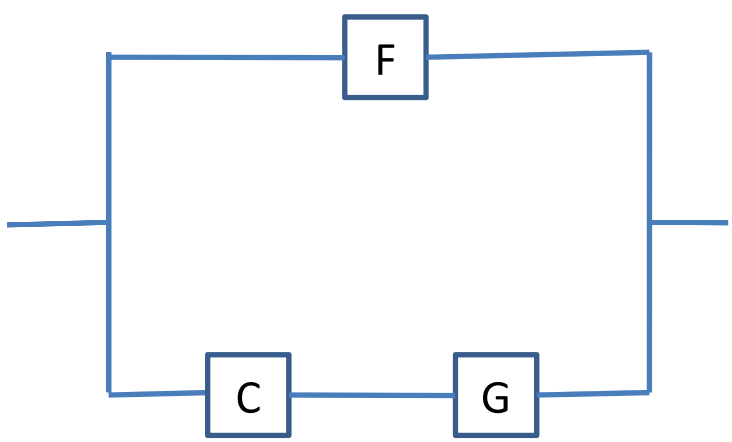
Time	t	0.1	0.2	0.5	0.7	1	1.5	2	3
Hazard function	$\lambda\alpha\, t^{\alpha-1}$	0.3795	0.5367	0.8485	1.0040	1.2000	1.4697	1.6971	2.0785
Reliability function	$\exp(-\lambda\, t^{\alpha})$	0.9750	0.9309	0.7536	0.6259	0.4493	0.2300	0.1041	0.0157
Density function	$\lambda\alpha\, t^{\alpha-1}\exp(-\lambda\, t^{\alpha})$	0.0250	0.0691	0.2464	0.3741	0.5507	0.7700	0.8959	0.9843
Distribution function	$1-\exp(-\lambda\, t^{\alpha})$	0.3700	0.4996	0.6395	0.6284	0.5392	0.3380	0.1766	0.0325



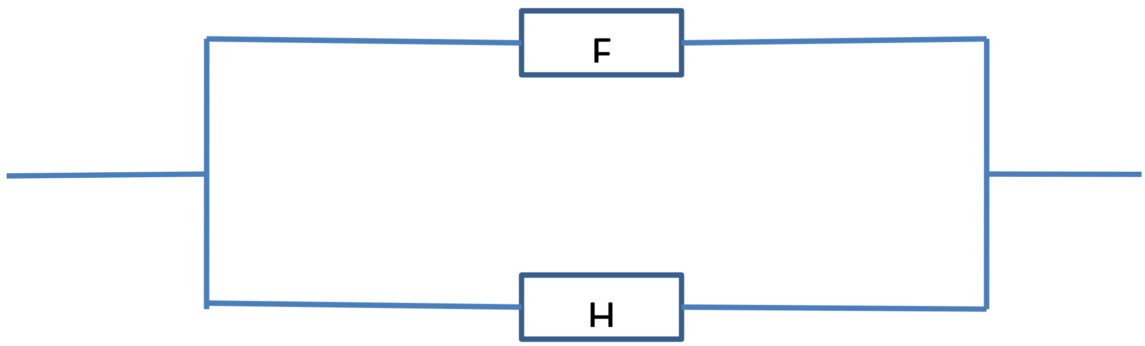
Example on parallel and series system: Each component in the system shown is opearable with probability 0.92 independently of other components. Calculate the reliability of the system:



a) The upper link A-B works if both A and B work. Thus we can replace this link with a component F that operates with probability $P(A \cap B) = (0.92)^2 = 0.8464$.
b) The components D and E connected in parallel can be replaced by component G that operates with probability $P(D \cup E) = 1 - (1 - 0.92)^2 = 0.9936$.
c) Now the configuration can be shown as



d) The components C and G connected in series so can be replaced by a component H that operates with probability $P(C \cap G) = (0.92)(0.9936) = 0.9141$. The new configuration can be as shown



e) Lastly, the components F and H are in parallel so the reliability of the system is

$P(F \cup H) = 1 - (1 - 0.8424)(1 - 0.9141) = 0.9868.$

Hence, reliability of the given system is 0.9868.

Conclusions:

We have studied various measures such as hazard function, reliability function, density function and distribution function of a life time of a component. Relationships among them are also seen. These are functions of time. Further, structure of these functions is different for different life time distributions. We have also studied how the reliability of each component contributes to the system reliability depending on the configuration of these components in the system. We have also learnt how to compute these numerical measurements.

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References:

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