# Actuarial Statistics - I

## Introduction

In this section, we will understand what are Actuarial Statistics and Utility Function and why utility function is necessary in the study of Insurance. Actuaries apply scientific principles and techniques from a range of other disciplines to problems involving risk, uncertainty and finance. Using mathematical and statistical methods actuarial science mainly involves in the study of reducing the risk or risk assessment.

"Utility" refers to the perceived value of a good and utility theory. For example, if you prefer vanilla ice-cream to chocolate, you would assign greater utility to vanilla ice-cream than to the same quantity of chocolate ice-cream. The fact that different agents have different utilities for goods is the basis of all markets. Mathematically we need a function to map between the physical measure of money and the perceived value of money. Such functions are utility functions, and in the context of wealth being a random variable on a probability space, utility functions are random variables.

### Introduction to Actuarial Statistics

An actuary applies mathematical and statistical techniques to financial problems. Actuaries manage financial risk and make financial sense of the future for their clients. They look at what's happened in the past and use it to make predictions about the future, developing appropriate strategies for the risks involved. The course is designed to give an understanding of sound practical ways in which financial risk can be managed.

Actuarial science would be particularly relevant are in the running of insurance companies and pension funds. The deeper understanding of applying the techniques and developed by looking in more detail at subjects in investment, insurance companies, and pension funds. In return for investing people's money, a promise has been made to pay out some benefits in the future. Actuarial science is needed to deal with the uncertainty of how much will need to be paid (amount) and when (timing). The amount and timing variables can be modelled mathematically to produce a workable model of the monetary liability today. The statistical distributions and methods will vary widely depending on the nature of the liability - modelling car accidents, latest industrial disease claims, and how long someone will need a pension to be paid for all different, and also carry varying economic cost.

Newer examples of applying actuarial science are in the assessment of capital projects and in helping a broad range of large financial organisations to better understand their liabilities and cater for them. There are a surprising number of complex assets held to protect and stabilise the company's finances.

Study routes will tend to begin with a thorough grounding of the mathematical, statistical and economic techniques which make up the foundations of actuarial science. By the end of this stage, a very good knowledge of business and finance will have been built up as well.

Because of the responsibilities which actuaries are called upon to undertake, they require to pass certain courses which are very limited in India. As there is a lot of demand for such professionals it is needed to be started at Universities

An insurance system is a mechanism for reducing the adverse financial impact of random events that prevent the fulfillment of reasonable expectations. There is a difference between insurance and related financial system like banking. Banking institutions were developed for the purpose of receiving, investing, and dispensing the savings of individuals and corporations. The cash flows in and out of a savings institution do not follow deterministic paths. In case of bank deposits, the returns are determined by what is totally paid until then, whereas in case of insurance, the risk is covered. In the event of loss of or other items, insurance company pays as assured. In financial investments payments are based on the size of a financial loss or profit occurring and are not associated with the person suffering the loss.

**Utility Theory**

"Utility" is the perceived ability of something to satisfy needs or wants. It is a representation of preferences over some set of goods and services. Utility is applied to generate an individual's value for the wealth. Utility theory is used in Economics, Finance, Statistics and many other subjects. For example, if you prefer mango to orange, you would assign greater utility to mango than to the same quantity of chocolate orange. In different subjects utilities have different basis for marking. In the Actuarial Science, the focus is on the utility of money. Mathematically we need a function to map between the physical measure of money and the perceived value of money. Such functions are called utility functions, and in the context of wealth being a random variable on a probability space, they need to be measurable functions on that space, and hence, utility function is a random variable X. A utility function, $u(x)$, can be described as a function which measures the value, or utility, that an individual (or institution) attaches to the monetary amount $x$. Usefulness of money is measured through utility function. The usefulness of Rs $x$ is $u(x)$, the *utility* (or ''moral value'') of Rs $x$. Typically, $x$ is the wealth or a gain of a decision-maker.

The insurance industry exists because people are willing to pay a price for being insured. There is an economic theory that explains why insureds are willing to pay a premium . A decision maker, generally without being aware of it, attaches a value $u(w)$ to his wealth $w$ instead of just $w$, where $u(\cdot)$ is called his *utility function*.

We suppose that a utility function $u(x)$ has two basic properties:(i) $u(x)$ is an increasing function of $x$, (ii) $u(x)$ is a concave function of $x$. Usually we assume that the function $u(x)$ is twice differentiable; then (i) and (ii) state that $u'(x) > 0$ and $u''(x) < 0$. The first property amounts to the evident requirement that more is better. Several reasons are given for the second property. One way to justify it is to require that the marginal utility $u'(x)$ be a decreasing function of wealth $x$, or equivalently, that the gain of utility resulting from a monetary gain of Rs $g$, $u(x + g) - u(x)$, be a decreasing function of wealth $x$.

The problem of presence of uncertainty will make decision taking difficult. One of the solutions is to define the value of an economic project with a random outcome to be its expected value, E[X]. By this expected value principle the distribution of possible outcome may be replaced for decision purposes by a single number, the expected value of the random monetary outcomes. In the absence of knowledge of value of random value E[X] will be used. By this principle, a decision maker would be indifferent between assuming the random loss X and paying an amount E[X] in order to be relived of possible loss. The expected value of random prospects with monetary payments is also called fair or actuarial value of the prospect.

To decide between random losses $X$ and $Y$, one compares $E[u(w-X)]$ with $E[u(w-Y)]$ and chooses the loss with the highest expected utility. With this model, the insured with wealth $w$ is able to determine the maximum premium, $P^+$ is prepared to pay for a random loss $X$. This is done by solving the equilibrium equation $E[u(w-X)] = u(w-P)$. At the equilibrium, he does not care, in terms of utility, if he is insured or not. The model applies to the other party involved as well. The insurer, with his own utility function and perhaps supplementary expenses, will determine a minimum premium $P^-$. If the insured's maximum premium $P^+$ is larger than the insurer's minimum premium $P^-$, both parties involved increase their utility if the premium is between $P^-$ and $P^+$.

Suppose that an insurance organization (insurer) was established to help reduce the financial consequences of damage or destruction of property. The insurer would issue contracts (policies) that would promise to pay the owner of the property a defined amount equal to or less than the financial loss if were damaged or destroyed during the period of the policy. The contingent payment linked to the amount of the loss is called a claim payment. In return for the promise contained in the policy, the owner of the property (issuered) pays a consideration (premium).

Imagine that an individual runs the risk of losing an amount $B$ with probability 0.05. He can insure himself against this loss, and is willing to pay a premium $P$ for this insurance policy. If $B$ is very small, then $P$ will be hardly larger than $0.05B$. However, if $B$ is somewhat larger, say 5000, then $P$ will be a little larger than 250. If $B$ is very large, $P$ will be a lot larger than $0.05B$, since this loss could result in bankruptcy. So the premium for a risk is not *homogeneous*, that is, not proportional to the risk.

Let us consider a decision problem faced by the owner of the property subject to loss. The property owner has a utility of wealth function u(w) where wealth w is measured in monetary terms. The owner faces a possible loss due to random events that may damage the property. The distribution of the random loss X is assumed to be known. The idea is that we assume that the owner will be indifferent between paying an amount G to the issuer, who will assume the random financial loss, and assuming the risk himself. The situation can be stated as

$$u(w-G) = E[u(w-X)].$$

Here, right-hand side is the expected utility when insurance is not purchased. That is on an average utility of what may remain from the wealth after the loss. Whereas left-hand side is the utility after paying G for insurance and the loss will be nullified or protected.

**Example:**

We have a utility function ranging from -1 to 0 with $u(0) = -1$ and $u(2{,}00{,}000) = 0$. Suppose one faces loss of Rs 2,00,000 with probability 0.5, and will remain at his current level of wealth with probability 0.5. What is the maximum amount $G$ he would be willing to pay for complete insurance protection against this random loss? This question can be expressed in the following way: For what value of $G$ does

$$u(2{,}00{,}000-G) = 0.5\, u(2{,}00{,}000) + 0.5\, u(0)$$

$$= 0.5(0) + 0.5\,(-1) = -\,0.5?$$

If he pays amount $G$, his wealth will certainly remain at $2{,}00{,}000-G$. The equal sign indicates that the decision maker is indifferent between paying $G$ with certainty and accepting the expected utility of wealth expressed on the right side. Suppose the decision maker's choice for $G = 1{,}20{,}000$. Then $u(2{,}00{,}000-1{,}20{,}000) = u(80{,}000) = 0.5$.

Normally, decision maker's response is that he is willing to pay an amount for insurance that is greater than the expected value of the loss, that is, $0.5(0) + 0.5(2{,}00{,}000) = 1{,}00{,}000$. Thus we have obtained a point for utility function at 80,000. Similarly, we can get utility function at another point. Once a utility value has been assigned to wealth levels $w_1$ and $w_2$, where $0 \le w_1 < w_2 \le 2{,}00{,}000$, we can determine an additional point by asking the decision maker the question "What is the maximum amount you would pay for complete insurance against the situation that could leave you with wealth $w_2$ with specified probability $p$, or at reduced wealth $w_1$ with probability $1-p$?". That is we asking the decision maker to fix a value G such that $u(w_2-G) = (1-p)\, u(w_1) + p\, u(w_2)$. This way we may able to get many utility values. Finally, based on these values a smooth function is estimated.

Within the range of financial outcomes for an individual insurance policy, the insurer's utility function might be approximated by a straight line. In this case, the insurer would adopt the expected value principle in setting its premium, as indicated above. The insurer would set its basic price for full coverage as the expected loss, $E[X]=\mu$. In this context $\mu$ is called the pure or net premium for the one period insurance policy. In practice this cannot be used as there will be expenses, taxes, profit and some security against adverse loss experience, the insurance system would decide to set the policy by loading , adding to the pure premium. For instance it might be $H = (1+a)\mu + c \quad a > 0,\ c > 0$.

Here $a\,\mu$ can be viewed as being associated with expenses that vary with expected losses and with the risk that claims experience will deviate from expected. The constant c provides expected expenses that do not vary with losses.

It is natural to assume that $u(w)$ is an increasing function, "more is better". In addition, it has been observed that for many decision makers, each additional equal increment of wealth results in a smaller increment of associated utility. This is called decreasing marginal utility in economics, that is $u''(x) < 0$ .

We assume that a utility function satisfies the conditions

$$u'(x) > 0 \quad \text{and} \quad u''(x) < 0. \tag{1}$$

Mathematically, the first of these conditions says that $u$ is an increasing function, while the second says that $u$ is a concave function. Simply put, the first states that an individual whose utility function is $u$ prefers amount $y$ to amount $z$ provided that $y > z$, that is the individual prefers more money to less! The second states that as the individual's wealth increases, the individual places less value on a fixed increase in wealth. For example, an increase in wealth of 1000 is worth less to the individual if the individual's wealth is 2,00,000 compared to the case when the individual's wealth is 1,00,000.

An individual whose utility function satisfies the conditions in (1) is said to be risk averse, and risk aversion can be quantified through the coefficient of risk aversion defined by

$$r(x) = -u''(x)/u'(x) \tag{2}$$

Utility theory can be used to explain why individuals are prepared to buy insurance, and to pay premiums which, by some criteria at least, are unfair. To illustrate why this is the case, consider the following situation. Most homeowners insure their homes against events such as fire on an annual basis. Although the risk of a home being destroyed by a fire in any year may be considered to be very small, the financial consequences of losing a home and all its contents in a fire could be devastating for a homeowner. Consequently, a homeowner may choose to pay a premium to an insurance company for insurance cover as the homeowner prefers a small certain loss (the premium) to the large loss that would occur if their home was destroyed, even though the probability of this event may be small. Indeed, a homeowner's preferences may be such that paying a premium that is larger than the expected loss may be preferable to not effecting insurance.

Many families of suitable utility functions exist that have interesting properties:
*linear utility*: $u(w) = w$
*quadratic utility*: $u(w) = -(\alpha-w)^2$ for $w \le \alpha$
*logarithmic utility*: $u(w) = \log(\alpha + w)$ for $(w > -\alpha)$
*exponential utility*: $u(w) = -\alpha\, e^{-\alpha w}$ $(\alpha > 0)$
*power utility*: $u(w) = w^c$ $(w > 0,\ 0 < c \le 1)$

## Practical 1: Computation of values of utility function.

(1) A decision makers utility function is given by $u(w) = \sqrt{w}$. The decision maker has wealth of $w$ =100 and faces a random loss $X$ with uniform distribution (0, 100). What is the maximum amount of this decision maker will pay for complete insurance against the random loss?

Solution:

Using the relation $u(w-G) = E[u(w-X)]$ we have

$$\sqrt{100 - G} = E[\sqrt{100 - X}]$$

$$= \frac{1}{100}\int_0^{100} \sqrt{100 - x}\ dx = \frac{2}{3}\frac{1}{100}\ 100^{3/2}$$

$$= 6.66667$$

$$100 - G = (6.66666667)^2 = 44.444444,$$

Hence, $G = 55.55555556 > 50 = E[X]$.

The decision maker is risk averse and $u'(x) > 0$. As expected $G > E[X] = 50$.