Factorial Experiment

In this lecture we discuss some concepts on Factorial experiment. The entire topic is subdivided in to the following subdivisions:

1. Basic definition and Principles
2. Advantages of Factorial experiment
3. 2^2 factorial experiment
4. Main and interaction effects and their estimate of 2^2 design
5. 2^3 factorial experiment.
6. Main and interaction effects and their estimate of 2^3 design
7. Conclusion

1. Basic definition and Principles

In simple trials , different fertilizers, several feeds of animals, different variety of seed, etc. are considered as treatments under investigation. In many experiments our interest confine in the effect of a factor at various levels and effect of the levels of a factor at different levels of other factors. The experiments, in which the effect of a number of levels of a factor is to be assessed with levels of other factors simultaneously are called factorial experiments. In factorial experiments, combinations of two or more levels of several factors is considered as treatments. These treatments are useful in studying the effect of each of the factors and the combination effect.

Example: Let us consider two fertilizers say Potash and Nitrogen. Let us suppose that there are p levels of Potash and q levels of Nitrogen. To find the effectiveness of various treatments (i.e. different levels of Potash and different levels of Nitrogen) we must conduct two simple experiments. One for Potash and one for Nitrogen. Such experiments do not give us information regarding the dependence or independence of one factor on the other. The other alternative to investigate the variations in several factors simultaneously is to conduct p X q factorial experiment.

Let us understand the utility of factorial experiments

Definitions.

A *factor* is an input of an experiment, and the number of forms the factor can take are called the *levels* of the factor.

A given experiment may include many factors. A *treatment* is a particular combination of one level from each factor.

If all, or nearly all, of the possible different combinations of treatment levels are of to be studied then the experiment is called a *factorial* experiment.

2. Advantages of Factorial experiment

The main advantages of factorial experiments, compared with the one factor at a time approach, are that

There is increased precision in the estimation of factor effects.

Interactions between factors can be explored.

Factorial designs allow the effect of a factor to be estimated at several levels of the other factors

The validity of conclusions can be tested fairly easily by inserting more factors.

3. 2^2 factorial experiment

2^2factorial experiment: (Factorial experiments with factors at two levels)

Example: The values of current and voltage in an experiment affect the rotation per minutes (*rpm*) of a fan speed. Suppose there are two levels of current.

5 Ampere, call it as level 1 (C_0) and denote it as a_0 (low level)

10 Ampere, call it as level 2 (C_1) and denote it as a_1 (High level)

Similarly, the two levels of voltage are

200 volts, call it as level 1 (V_0) and denote it as b_0 (low level)

220volts, call it as level 2 (V_1) and denote it as b_1 (High level)

The two factors are denoted as *A*, say for current and *B*, say for voltage.

In order to make an experiment, there are 4 different combinations of values of current and voltage

5 Ampere Current and 200 Volts Voltage, denoted as $C_oV_o = a_0b_o$

2. 5 Ampere Current and 220 Volts Voltage, denoted as $C_oV_1 = a_0b_1$
3. 10 Ampere Current and 200Volts Voltage, denoted as $C_1V_o = a_1b_o$
4. 10 Ampere Current and 220 Volts Voltage, denoted as $C_1V_1 = a_1b_1$

The responses from those treatment combinations are represented by $a_0b_0 = (1)$, $a_0b_1 = (b)$, $a_1b_0=(a)$ and $a_1b_1 = (b)$, Respectively.

4. Main and interaction effects and their estimate of 2^2 design

Consider the following:

Main effect A:

The effect of current for Voltage level V_0 is defined as $C_1V_0 - C_0V_0 = (a)-(1)$

The effect of current for Voltage level V_1 is defined as $C_1V_1 - C_0V_1 = (ab)-(b)$

The two effects are termed as simple effects of factor A

Averaging these two quantities yield main effect of Current

$= \dfrac{(ab)-(b)+(a)-(1)}{2}$

$= \dfrac{1}{2}\{(ab)-(b)+(a)-(1)\}$ or **main effect of Factor A**

Main effect B:

The effect of voltage for current level c_0 is defined as $C_0V_1 - C_0V_0 = (b)-(1)$

The effect of voltage for current level c_1 is defined as $C_1V_1 - C_1V_0 = (ab)-(a)$

The two effects are termed as simple effects of factor A.

Averaging these two quantities yield main effect of Voltage

I.e. B$= \dfrac{(ab)-(a)+(b)-(1)}{2}$

$= \dfrac{1}{2}\{(ab)+(b)-(a)-(1)\}$ or **main effect of Factor B**

Interaction effect AB

Average difference between the effect of A at the high level of B and the effect of A at low level of B is the interaction effect AB

I.e. AB $=½\{(ab)-(b)$ -$[(a)-(1)]\} = \dfrac{1}{2}\{(ab)-(b)-(a)+(1)\}$

Alternatively AB also defined as the average difference between the effect of B at high level of A and the effect of B at low level of A

I.e. AB =1/2{$(ab)-(a)$ -[$(b)-(1)$]} = $\frac{1}{2}\{(ab)-(b)-(a)+(1)\}$

Mean effect (M):

The quantity $\frac{C_0V_0+C_1V_0+C_0V_1+C_1V_1}{4}$ = $\frac{1}{4}\{(ab)+(b)+(a)+(1)\}$

Gives mean effect of all treatment combinations.

Treating (ab) as (a) (b) symbolically (Mathematically and conceptually it is incorrect), we can now express all the main effects, interaction effects as follows:

Main effect A = $\frac{1}{2}\{(ab)-(b)+(a)-(1)\}$ = $\frac{(a-1)(b+1)}{2}$

Main effect B = $\frac{1}{2}\{(ab)+(b)-(a)-(1)\}$ = $\frac{(a+1)(b-1)}{2}$

Interaction effect AB = $\frac{1}{2}\{(ab)-(b)-(a)+(1)\}$ = $\frac{(a-1)(b-1)}{2}$

General mean effect (M) = $\frac{1}{4}\{(ab)+(b)+(a)+(1)\}$ = $\frac{(a+1)(b+1)}{2}$

Notice the roles of + and – signs as well as the divisor.

There are two effects related to *A* and *B*.

To obtain the effect of a factor, write the corresponding factor with – sign and others with + sign.

For example, in the main effect of *A*, *"a"* occurs with – sign as in (*a* -1) and *"b"* occurs with + sign as in (*b* +1).

In *AB*, both the effects are present so *"a"* and *"b"* both occur with + signs as in (*a*+1) (*b*+1).

Also note that the main and interaction effects are obtained by considering the typical differences of averages, so they have divisor 2 where as general mean effect is based on all the treatment combinations and so it has divisor 4.

These effects can be represented in the following table

Factorialeffects	Treatmentcombinations				Divisor
	(1)	(*a*)	(*b*)	(*ab*)	
M	+	+	+	+	4
A	-	+	-	+	2
B	-	-	+	+	2
AB	+	-	-	+	2

If r replications from each of the treatment combinations are obtained , then the expressions for the main and interaction effects can be expressed as:

main effect of Factor A = $\frac{1}{2r}\{(ab)-(b)+(a)-(1)\}$

main effect of Factor B = $\frac{1}{2r}\{(ab)+(b)-(a)-(1)\}$

Interaction effect AB = $\frac{1}{2r}\{(ab)-(b)-(a)+(1)$

Example: The following table gives the layout and results of a 2^2 factorial design laid out in four replicates. The purpose of the experiment is to determine the effect of different kind of fertilizers Nitrogen (N), Potash (K) on potato yield.

Bolck 1				Block 2			
1	n	k	nk	K	n	1	nk
261	391	312	373	301	265	206	450
Bolck 3				Block 4			
n	1	k	nk	nK	n	1	k
381	271	312	362	401	265	116	310

Compute the total yield on different treatment combinations as given below:

Treatment Combination					Total	
1	261	206	271	116	854	4977
n	391	265	381	265	1302	799
k	312	301	312	310	1235	665
nk	373	450	362	401	1586	-97

From the data above , we can estimate the average effect as(i.e. estimates of Main effects and interaction effects as) estimate

A = $\frac{1}{2r}\{(ab)-(b)+(a)-(1)$

= (1/2x3)[1297-1175+1182-864)= 799/6 = 133.17

B = $\frac{1}{2r}\{(ab)+(b)-(a)-(1)$

= (1/2x3)[1297+1175-1182-864)= 665/6 = 110.83

AB = $\frac{1}{2r}\{(ab)-(b)-(a)+(1)$

= (1/2x3)[1297-1175-1182+864) = -97/6= 16.17

Effect A is positive suggests that increasing A from low level to high level will increase the yield.

Effect B is positive suggests that increasing A from low level to high level will increase the yield

Effect AB is negative suggests that the interation effect is in reverse direction in increasing the yield (Increasing level of A with decreasing level of B will increase the yield or vise versa). Here Interaction effect appear to be small relative to the two main effects, which suggests that increasing level of main effects will increase the yield.

5.2^3 factorial experiment.

Suppose that in a complete factorial experiment , there are three factors - A, B and C each at two levels, viz -a0, a1; b0, b1; and c0, c1 respectively. This design is called 2^3 factorial experiment. There are total eight treatment combinations 1, a, b, ab, c, ac, bc and abc [1 denote the factors A,B, and C all are at lower level, a denote the factor A at higher level and B and C art at lower level, b denote the factor B at higher level and A and C art at lower level, and so on].

Main and interaction effects:

As in the case of 2^2 factorial experiment, the main effect of any factor is defined as the average effect of the particular factor fixing the other factor at a particular level.

6. Main and interaction effects and their estimate of 2^3 design

Main effect of factor A:

The effect of Factor A for factor B at lower level (b_0) and factor C at lower level (c_0) is defined as $A_1B_0C_0 - A_0B_0C_0$ = $(a)-(1)$

The effect of Factor A for factor B at higher level (b_1) and factor C at c_0 lower level (c_0) is defined as

$A_1B_1C_0 - A_0B_1C_0$ = $(ab)-(b)$

The effect of Factor A for factor B at lower level (b_0) and factor C at higher level (c_1) is defined as

$A_1B_0C_1 - A_0B_0C$ = $(ac)-(c)$

The effect of Factor A for factor B at high level (b_1) and factor C at high level (c_1) is defined as $A_1B_1C_1 - A_0B_1C_1$ = $(abc)-(bc)$

These four effects are termed as simple effects of A

Thus the average effect of A is just the average of these 4 simple effects

i.e. main effect of A is

A = $\frac{1}{4}[(a)-(1)+(ab)-(b)+(abc)-(bc)+(ac)-(c)]$

Similarly main effect of B is defined as

B = $\frac{1}{4}[(abc)+(bc)-(a)-(1)+(ab)+(b)-(ac)-(c)]$

And main effect of C is

C = $\frac{1}{4}[(abc)+(bc)-(a)-(1)-(ab)-(b)+(ac)+(c)]$

Interaction effect AB:

When C is at lower level, simple effects of A are $(a)-(1)$ and $(ab)-(b)$ therefore interaction AB is

$\frac{1}{2}[(ab)-(b)-[(a)-(1)]] = \frac{1}{2}[(ab)-(b)-(a)+(1)]$

When C is at higher level, simple effects of A are $(ac)-(c)$ and $(abc)-(bc)$ therefore interaction AB is

$\frac{1}{2}[(abc)-(bc)-[(ac)-(c)]] = \frac{1}{2}[(abc)-(bc)-(ac)+(c)]$

The average of the two interactions is termed as interaction AB in the presence of C

Therefore interaction effect AB is

AB = $\frac{1}{4}[[(ab)-(b)-(a)+(1)]+[(abc)-(bc)-(ac)+(c)]]$

Similarly interaction effect of BC and AC are respectively given by

AC = $\frac{1}{4}[[(abc)-(bc)+(ac)-(c)-(ab)+(b)-(a)+(1)]$

BC = $\frac{1}{4}[[(abc)+(bc)-(ac)-(c)-(ab)-(b)+(a)+(1)]$

Three factor interaction ABC:

Average difference between AB interaction at two different levels of C is the interaction effect ABC

i.e. ABC = $\frac{1}{4}[[(abc)-(bc)]-[(ac)-(c)]-[(ab)-(b)]+[(a)-(1)]$

= $\frac{1}{4}[(abc)-(bc)$-(ac)+(C)-(ab)+ (b)+(a)-(1)]

The effects are determined using the following table

Factorial effect	Treatment combinations								Divisor
	(1)	a	b	ab	c	ac	bc	abc	
I	+	+	+	+	+	+	+	+	8
A	-	+	-	+	-	+	-	+	4
B	-	-	+	+	-	-	+	+	4
AB	+	-	-	+	+	-	-	+	4
C	-	-	-	-	+	+	+	+	4
AC	+	-	+	-	-	+	-	+	4
BC	+	+	-	-	-	-	+	+	4
ABC	-	+	+	-	+	-	-	+	4

The role of + sign and – sign are also symbolically remembered as

Main effect A = $\frac{(a-1)(b+1)(c+1)}{4}$

Main effect B = $\frac{(a+1)(b-1)(c+1)}{4}$

Main effect C = $\frac{(a+1)(b+1)(c-1)}{4}$

Interaction effect AB = = $\frac{(a-1)(b-1)(c+1)}{4}$

Interaction effect AC = = $\frac{(a-1)(b+1)(c\_1)}{4}$

Interaction effect BC = = $\frac{(a+1)(b-1)(c-1)}{4}$

Interaction effect ABC = = $\frac{(a-1)(b-1)(c-1)}{4}$

General mean effect (M) = $\frac{(a+1)(b+1)(c+1)}{8}$

If r replication from each of the treatment combinations are obtained , then the expressions for the main and interaction effects can be expressed as

A = ¼r$[(a)-(1)+(ab)-(b)+(abc)-(bc)+(ac)-(c)]$

B = ¼r$[(abc)+(bc)-(a)-(1)+(ab)+(b)-(ac)-(c)]$

C = ¼r$[(abc)+(bc)-(a)-(1)-(ab)-(b)+(ac)+(c)]$

AB = ¼r$[[(ab)-(b)-(a)+(1)]+[(abc)-(bc)-(ac)+(c)]]$

AC = ¼r$[[(abc)-(bc)+(ac)-(c)-(ab)+(b)-(a)+(1)]$

BC = ¼r$[[(abc)+(bc)-(ac)-(c)-(ab)-(b)+(a)+(1)]$

ABC = ¼r[$(abc)-(bc)$-(ac)+(C)-(ab)+ (b)+(a)-(1)]

Example:

For a factorial experiment with three factors N, P and K each at two levels, the design and yield per plot are given below.

Replicate 1	Replicate 2	Replicate 3
np 30	1 44	pk 20
nk 32	nk 34	1 24
pk 24	p 27	npk 30
1 25	npk 36	k 32
n 46	k 32	n 28
k 39	n 30	p 26
p 32	np 30	np 36
npk 42	pk 36	nk 28

Calculate the total yield from each treatment combination as in the following table

Treatment Combination				Total
1	25	44	24	93
n	46	30	28	104
p	32	27	26	85
k	39	32	32	103
np	30	30	36	96
nk	32	34	28	94
pk	24	36	20	80
npk	42	36	30	108

From the data above the estimates of Main and interaction effects are given as

A = ¼r$[(a)-(1)+(ab)-(b)+(abc)-(bc)+(ac)-(c)]$ =

1/4x3[104-94+96-85+108-80+94-103]=40/12=3.42

B = ¼r$[(abc)+(bc)-(a)-(1)+(ab)+(b)-(ac)-(c)]$

1/4x3[108+80-104-94+96+85-94-103]=-26/12=-2.08

C = ¼r$[(abc)+(bc)-(a)-(1)-(ab)-(b)+(ac)+(c)]$

1/4x3[108+ 80- 104- 94- 96- 85+94+103]=-6/12= 0.58

AB = ¼r$[[(ab)-(b)-(a)+(1)]+[(abc)-(bc)-(ac)+(c)]]$= 38/12= 3.08

AC = ¼r$[[(abc)-(bc)+(ac)-(c)-(ab)+(b)-(a)+(1)]$ = -2/12 = -0.25

BC = ¼r$[[(abc)+(bc)-(ac)-(c)-(ab)-(b)+(a)+(1)]$ = -8/12 = 0.58

ABC = ¼r$[(abc)-(bc)$-(ac)+(C)-(ab)+ (b)+(a)-(1)] =36/12 = 3.08

Effect A is positive suggests that increasing amount of N will increase the yield.

Effect B is negative suggests that increasing amount of P will decrease the yield

The effect of K appears to be small compared to the two main effects , N and P

Interaction effect AB and AC are small as compared to other main and interaction effect..etc.

7. Conclusion:

In this lecture we discussed the basic definition of factorial experiments, their advantages. We discussed the 2^2 factorial experiments, discussed their main and interaction effects. We studied these factorial effects and their meaning and their estimates by considering a simple example. We also discussed 2^3 factorial experiment, the main and interaction effects of 2^3 design . Studied these concepts by taking an example.