2^3 Factorial Experiment

In this lecture we discuss the analysis of 2^3 Factorial experiment considered under RBD.

The entire topic is divided in to the following subdivision:

1. Model Statement :
2. Hypothesis
3. Test Statistics
4. Analysis of Variance Table
5. Yates technique to compute factorial effects:
6. Example
7. Conclusion

1. Model:

Suppose that 2^3 Factorial experiment is conducted in a RBD with r replicates (Blocks) .

The model under study is

$$y_{(x_1,x_2,x_3)j} = \mu + \alpha(x_1,x_2,x_3) + \beta_j + \varepsilon_{(x_1,x_2,x_3)j}, \quad (x_1, x_2, x_3: 0 \text{ or } 1),\ J = 1,2,\ldots r$$

Where

$y_{(x_1,x_2,x_3)j}$ - observations from j[th] replicate receiving the treatment combination(x_1, x_2, x_3), (x_1, x_2, x_3: 0 or 1)

μ = general effect

$\alpha(x_1, x_2, x_3)$ =effect of the treatment combination (x_1, x_2, x_3)

β_j = jth replicate effect

$\varepsilon_{(x_1,x_2,x_3)j}$ =error term

Let [A], [B], [C], [AB] , [AC], [BC] and [ABC] represent the total responses (Total factorial effects) and are defined as

[A] = [abc] - [bc] + [ac] – [c] + [ab] - [b] + [a] - [1]

[B] = [abc] + [bc] - [ac] – [c] + [ab] + [b] - [a] - [1]

[C] = [abc] + [bc] + [ac] + [c] - [ab] - [b] - [a] - [1]

[AB] = [abc] - [bc] - [ac] + [c] + [ab] - [b] - [a] + [1]

[AC] = [abc] - [bc] + [ac] – [c] - [ab] + [b] - [a] + [1]

[BC] = [abc] + [bc] - [ac] – [c] - [ab] - [b] + [a] + [1]

[ABC] = [abc] - [bc] - [ac] + [c] - [ab] + [b] + [a] - [1]

[abc] , [bc] , [ac] , [c] , [ab] , [b] , [a] and [1] represent the total yield of all the replicates receiving the treatment combinations **treatments (1), (*a*), (*b*), (c), (*ab*), (ac), (bc) and (abc)**

respectively.

2. **Hypothesis**

We test the following hypnosis.

1. H_a: A =0 (i.e.. Main effect of A is absent)

   Against the alternative

   H_a1: A #0 (i.e.. Main effect of A is present)
2. H_b: B =0 (i.e.. Main effect of B is absent)

   Against the alternative

   H_b1: B #0 (i.e.. Main effect of B is present)
3. H_c: C =0 (i.e.. Main effect of C is absent)

   Against the alternative

   H_c1: C #0 (i.e.. Main effect of C is present)
4. H_{ab}: AB =0 (i.e.. Interaction effect A B is absent)

   Against the alternative

   $H_{ab}1$: AB #0 (i.e.. Interaction effect A B is present)
5. H_{ac}: AC =0 (i.e.. Interaction effect A C is absent)

   Against the alternative

   $H_{ac}1$:AC #0 (i.e.. Interaction effect A C is present)
6. H_{bc}: BC =0 (i.e.. Interaction effect BC is absent)

   Against the alternative

   $H_{bc}1$: BC #0 (i.e.. Interaction effect BC is present)
7. H_{abc}: ABC =0 (i.e.. Three factor Interaction effect A BC is absent)

   Against the alternative

   $H_{abc}1$: A BC#0 (i.e.. Three factor Interaction effect A BC is present)

   We can also test simultaneously
8. H_β: $\beta_1 = \beta_2 = \beta_3 \ldots\ldots = \beta_r = 0$

   The replication effect is insignificant.

3. **Test Statistics**

The test statistics to test the above hypothesis are respectively

$F_A = \frac{[A]^2/8r}{ESS/7(r-1)}$ which follow F distribution with 1 and 37(r-1) degree of freedom

$F_B = \frac{[B]^2/8r}{ESS/7(r-1)}$ which follow F distribution with 1 and 7(r-1) degree of freedom

$F_C = \frac{[C]^2/8r}{ESS/7(r-1)}$ which follow F distribution with 1 and 7(r-1) degree of freedom

$F_{AB} = \frac{[AB]^2/8r}{ESS/7(r-1)}$ which follow F distribution with 1 and 7(r-1) degree of freedom

$F_{AC} = \frac{[AC]^2/8r}{ESS/7(r-1)}$ which follow F distribution with 1 and 7(r-1) degree of freedom

$F_{BC} = \frac{[BC]^2/8r}{ESS/7(r-1)}$ which follow F distribution with 1 and 7(r-1) degree of freedom

$F_{ABC} = \frac{[ABC]^2/8r}{ESS/7(r-1)}$ which follow F distribution with 1 and 7(r-1) degree of freedom

$F_\beta = \frac{BSS/r-1}{ESS/7(r-1)}$ which follow F distribution with r-1 and 3(r-1) degree of freedom

The expression for sum of squares due to error is given by ESS= TSS-TRSS-BSS,

Sum of square Due to total is computed as $TSS = \sum_{(x1,x2,X_3)=0}^{1} \sum_{k=1}^{r} y^2_{(x1,x2,X_3)k} - \frac{G^2}{8r}$

And $G = \sum_{(x1,x2,X_3)=0}^{1} \sum_{k=1}^{r} y_{(x1,x2,X_3)k}$

Sum of square due to replicate is computed as $BSS = \frac{\sum_{k=1}^{r} Y^2_{(...)k}}{8} - \frac{G^2}{8r}$ where $y_{(..)k}$ is the kth replicate total

TRSS = Sum of squares due to A+Sum of squares due to B+ Sum of squares due to C + Sum of squares due to AB + Sum of squares due to AC + Sum of squares due to BC + Sum of squares due to ABC

And is computed as

TRSS = $[A]^2/8r$ + $[B]^2/8r$ + $[C]^2/8r$ + $[AB]^2/8r$ + $[AC]^2/8r$ + $[BC]^2/8r$ + $[ABC]^2/8r$

4. Analysis of Variance Table

Source of Variation	**Degree of freedom**	**Sum of squares**	**Mean sum of squares**	**F ratio**	**Critical value**
Main factor A	1	SSA = $[A]^2/8r$	*MSSA=SSA/1*	$F_A = \frac{[A]^2/8r}{MESS}$	$F_{(1\ ,\ 7(r-1))}$

Main factor B	1	SSB = $[B]^2/8r$	MSSB=SSB/1	$F_B = \frac{[B]^2/8r}{MESS}$	$F_{(1, 7(r-1))}$
Main factor C	1	SSC = $[C]^2/8r$	MSSC=SSC/1	$F_C = \frac{[C]^2/8r}{MESS}$	$F_{(1, 7(r-1))}$
Interaction AB	1	SSAB = $[AB]^2/8r$	MSSAB= SSAB/1	$F_{AB} = \frac{[AB]^2/8r}{MESS}$	$F_{(1, 7(r-1))}$
Interaction AC	1	SSAC = $[AC]^2/8r$	MSSAC=SSAC/1	$F_{AC} = \frac{[AC]^2/8r}{MESS}$	$F_{(1, 7(r-1))}$
Interaction BC	1	SSBC = $[BC]^2/8r$	MSSBC=SSBC/1	$F_{BC} = \frac{[BC]^2/8r}{MESS}$	$F_{(1, 7(r-1))}$
Interaction ABC	1	SSABC = $[ABC]^2/8r$	MSSABC=SSABC/1	$F_{ABC} = \frac{[ABC]^2/}{MESS}$	$F_{(1, 7(r-1))}$
Replicate	r-1	$BSS = \frac{\sum_{k=1}^{r} Y^2_{(...)k}}{8} - \frac{G^2}{8r}$	MBSS= BSS/(r-1)	MBSS/MESS	$F_{(r-1, 7(r-1))}$
Error	7(r-1)	ESS	MESS= ESS/7(r-1)		
Total	8r-1	$TSS = \sum_{(x1,x2,X_3)=0}^{1} \sum_{k=1}^{r} y^2_{(x1,x2,X_3)k} - \frac{G}{8r}$			

Conclusion : The decision rule is to reject the concerned null hypothesis when the value of concerned F statistics

$F_{calculated} > F_{(1, 7(r-1))}$ and for replicate $F_{calculated} > F_{(r-1, 7(r-1))}$ at α significance level i.e.

If the calculated values of the test statistics exceeds the table value (Critical) Value, we reject the hypothesis. Otherwise accept it.

5. Yates technique to compute factorial effects:

Step1 1: Write the treatment combination in the first column in the order, (1), (a), (b),(ab), (c), (ac), (bc) and (abc)

Step 2: Against each treatment combination, in the second column write the corresponding total yield from all the replicates.

Step 3: The entries in the third column are as follows: the first half entries are obtained by writing down in order, the pairwise sums of the values in column 2 and the second half is obtained by writing in the same order the pairwise differences of the values in the second column, the difference is taken such that the first number a pair is subtracted from the second number of the pair.

Step 4: The whole of the procedure explained in step 3 is repeated on the 3rds column and 4th column is derived.

Step 5: The procedure explained in step 3 is repeated on the 4th column and 5th column is derived

The first entry in the last column gives the grand total, the other entries in the last column are the total of the main effects and interaction effects corresponding to the treatment combinations denoted in the first column of the table.

The entire procedure is explained in the following table

Treatment Combination	Total yield	Column 3	Column 4	Factorial effects	Factorial effects	Best estimates
1	[1]	[1]+ [a]	[1]+ [a]+ [b]+ [ab]	[1]+ [a]+ [b]+ [ab]+ [c]+ [ab]+ [bc]+ [abc	Grand Total= G	
a	[a]	[b]+ [ab]	[c]+ [ab]+ [bc]+ [abc	[a]- [1]+ [ab]- [b]+ [ab]-[c]+ [abc]-[bc]	[A]	[A]/4r
b	[b]	[c]+ [ab]	[a]- [1]+ [ab]- [b]	[b]+ [ab]-{ [1]+ [a]}+ [bc]+ [abc-{[c]+ [ab]}	[B]	[B]/4r
ab	[ab]	[bc]+ [abc	[ab]-[c]+ [abc]-[bc]	[ab]- [b]-{ [a]- [1]}+ [abc]- [bc]-{ [ab]-[c]}	[AB]	[AB]/4r
c	[c]	[a]- [1]	[b]+ [ab]-{ [1]+ [a]}	[c]+ [ab]+ [bc]+ [abc]-{ [1]+ [a]+ [b]+ [ab]}	[C]	[C]/4r

ac	[ab]	[ab]- [b]	[bc]+ [abc-{[c]+ [ab]}	[ab]-[c]+ [abc]-[bc]-{ [a]- [1]+ [ab]- [b]}	[AC]	[AC]/4r
bc	[bc]	[ab]-[c]	[ab]- [b]-{ [a]- [1]}	[bc]+ [abc-{[c]+ [ab]}-{ [b]+ [ab]-{ [1]+ [a]}}	[BC]	[BC]/4r
abc	[abc]	[abc]-[bc]	[abc]-[bc]-{ [ab]-[c]}	[abc]-[bc]-{ [ab]-[c]}-{ [ab]-[b]-{ [a]- [1]}}	[ABC]	[ABC]/4r

[The order of selection of treatment combination is very important in the yates table as explained below: Starting with the treatment combination (1) , each factor introduced in turn and is then followed by all combinations of itself with . Hence in the case of 2^3 factorial experiment, Introduce (1), followed by the factor (a) then introduce to 'b' to (1) ans (a) hence we get (b), (ab). Then introduce c to the above combinations hence we get, (c), (ac), (bc) and (abc)

Example: For a factorial experiment with three factors N, P and K each at two levels, the design and yield per plot are given below.

<u>Replicate 1</u>	<u>Replicate 2</u>	<u>Replicate 3</u>
np 30	1 44	pk 20
nk 32	nk 34	1 24
pk 24	p 27	npk 30
1 25	npk 36	k 32
n 46	k 32	n 28
k 39	n 30	p 26
p 32	np 30	np 36
npk 42	pk 36	nk 28

Analyse the data.

Solution

Model: We define the model

$y_{(x_1,x_2,x_3)j} = \mu + \alpha(x_1,x_2,x_3) + \beta_j + \varepsilon_{(x_1,x_2,x_3)j}$, *($x_1$, x_2, x_3: 0 or 1), J=1,2,….3*

, J=1,2,….r

Hypothesis:

1. H_a: N =0 (i.e.. Main effect of A is absent)

   Against the alternative

   H_a1: N #0 (i.e.. Main effect of N is present)

2. H_b: P =0 (i.e.. Main effect of P is absent)

   Against the alternative

   H_b1: P #0 (i.e.. Main effect of P is present)

3. H_c: K =0 (i.e.. Main effect of K is absent)

   Against the alternative

   H_c1: K #0 (i.e.. Main effect of C is present)

4. H_{ab}: NP =0 (i.e.. Interaction effect NP is absent)

   Against the alternative

   $H_{ab}1$: NP #0 (i.e.. Interaction effect NP is present)

5. H_{ac}: NK =0 (i.e.. Interaction effect NK is absent)

   Against the alternative

   $H_{ac}1$:NK #0 (i.e.. Interaction effect NK is present)

6. H_{bc}: PK =0 (i.e.. Interaction effect PK is absent)

   Against the alternative

   $H_{bc}1$: PK #0 (i.e.. Interaction effect PK is present)

7. H_{abc}: NPK =0 (i.e.. Three factor Interaction effect NPK is absent)

   Against the alternative

   $H_{abc}1$: NPK#0 (i.e.. Three factor Interaction effect NPK is present)

   We can also test simultaneously

Computation:

Yates table:

Treatment Combination	Total yield	Cloumn 3	Column 4	Factorial efects	Sum of squares	Best estimates
1	93	197	378	Grand Total = 763		
n	104	181	385	[N] = 41	70.04167	3.42
P	85	197	22	[P] =-25	26.04167	-2.08
np	96	188	19	[NP] =37	57.04167	3.08
k	103	11	-16	[K] =7	2.041667	0.580
nk	94	11	-9	[NK] =-3	0.375	-0.25
pk	80	-9	0	[PK] =7	2.041667	0.58
npk	108	28	37	[NPK] =37	57.04167	3.08

$$TSS = \sum_{(x1,x2,X_3)=0}^{1} \sum_{k=1}^{r} y^2_{(x1,x2,X_3)k} - \frac{G^2}{8r} = 25227 - 763^2/24 = 969.04$$

BSS = $195437/8-(763^2/24) = 172.58$

[Each replicate total is 270, 269, 224]

Total of Sum of squares due to all factorial effects = 70.04167+26.04167+…+57.04167 = 214.625

ESS = 969.04- 172.58-214.625 = 582.75

=24783.56

Analysis of Variance Table.

Source of Variation	Degree of freedom	Sum of squares	Mean sum of squares	F ratio	Critical value
Main factor N	1	SSN = =70.04167	70.04	1.68	$F_{(1\ ,\ 14)}$= 4.60
Main factor P	1	SSP = =26.04167	26.04	0.63	4.60
Main factor K	1	SSK =2.041667	2.04	0.049	4.60
Interaction NP	1	SSNP = 57.0416	57.04	1.37	4.60
Interaction NK	1	SSNK = 0.375	0.375	9.00	4.60
Interaction PK	1	SSPK = 2.041667	2.04	0.049	4.60
Interaction NPK	1	SSNPK = 57.04167	57.04	1.37	4.60
Replicate	r-1=2	$BSS = \frac{\sum_{k=1}^{r} Y^2_{(...)k}}{8} - \frac{G^2}{8r}$ =172.58	*86.29*	2.07	$F_{(2,14)}$=3.74
Error	7(r-1)=14	ESS=582.75	41.625		
Total	23	$TSS = \sum_{(x1,x2)=0}^{1} \sum_{k=1}^{r} y^2_{(x1,x2)k} - \frac{G^2}{4r}$ =969.96			

From the table observe that the we do not reject any of the hypotheses 1 to 7 and all the i.e. main effects and interaction effects are absent in the data(Though we observed positive or negative effects these are not significant) . We can also observe that the replicate effects are insignificant.

Conclusion: In this session we discussed the RBD model under study for2^3 factorial experiment. We stated the different hypothesis to be tested under this model. We defined the test statistics used to test the stated hypothesis. We summarized the entire analysis under ANOVA table. We discussed the method of computing factorial effect totals using Yates method. Considering an example under RBD we discussed the analysis of 2^3 Factorial experiment.