1. Introduction

Welcome to the series of E learning modules on LRTP for testing equality of two means of two univariate Normal distributions. In this module we are going cover the Likelihood Ratio Test Procedure for testing the difference of means of two Normal populations when the variances are known and unknown and also a test criterion for pair wise testing of the independent samples drawn from the Normal population.

By the end of this session, you will be able to:

- Explain LRTP to test the difference between the means of two Normal populations when the variances are known and unknown
- Explain LRTP to test the difference between means in case of dependent or correlated observations.

In this paper let us discuss about the LRTP available for testing means of two independent Normal populations based on the fixed variance or an unknown variance and also for two dependent samples of a Normal population.

2. Application 1

Application one:
To test the null hypothesis H naught: mu one is equal to mu two against the alternative hypothesis H one: mu one is not equal to mu two. Where mu one and mu two are the means of two Normal populations with a common variance sigma square which is unknown.

Let x one, x two, etc till xn be a random sample of size 'n' from a Normal population with mean mu one and let y one, y two, etc till YM be a random sample of size M from a Normal population with mean mu two. Then the likelihood ratio lambda is given by:
lambda is equal to Supremum of L of (x one, x two, etc till xn, y one, y two, etc till YM) when mu one is equal to mu two is equal to mu and sigma square divided by Supremum of L of (x one, x two, etc till xn, y one, y two, etc till YM) under mu one coma mu two coma sigma square.

Lambda is equal to Supremum of L of (x one, x two, etc till xn) into L(y one, y two, etc till YM) when mu one is equal to mu two is equal to mu and sigma square divided by Supremum of L of (x one, x two, etc till xn) into L(y one, y two, etc till YM) under mu one, mu two coma sigma square.
Since all the observations are independent of each other, lambda is equal to Supremum of (one by sigma into square root two pi) to the power 'n' into 'e' to the power minus one by two sigma square into summation (xi minus mu one) the whole square into (one by sigma into square root of two pi) to the power 'm' into 'e' to the power minus one by two sigma square into summation (yi minus mu two) the whole square, when mu one is equal to mu two is equal to mu and sigma square.
Divided by Supremum of (one by sigma into square root into two pi)to the power 'n' into 'e' to the power minus one by two into sigma square into summation (xi minus mu one) the whole square into (one by sigma into square root of two pi) to the power 'm' into 'e' to the power minus one by two into sigma square into summation (yi minus mu two) the whole square under mu one, mu two coma sigma square.

Under the null hypothesis the m.l.e's are obtained as mu cap is equal to 'n' x bar plus 'm' y bar by 'm' plus 'n' and sigma naught square is equal to summation (xi minus mu cap) the whole square plus summation (yi minus mu cap) the whole square by 'm' plus 'n'
Otherwise the m.l.e's are mu one cap is equal to x bar, mu two cap is equal to 'y' bar and sigma cap square is equal to summation (xi minus x bar) the whole square plus summation (yi minus y bar) the whole square by 'm' plus 'n'

Thus substituting the m.l.e's appropriately, we have lambda is equal to (one by sigma cap naught square) to the power (m plus n) by two into (one by square root of two pi) to the power(m plus n)into 'e' to the power minus one by two sigma cap naught square into summation (xi minus mu cap) the whole square plus summation (yi minus mu cap) the whole square.
Divided by (one by sigma cap square) to the power (m plus n) by two into (one by square root of two pi) to the power (n plus m) into 'e' to the power minus one by two sigma cap square into summation (xi minus x bar) the whole square plus summation(yi minus y bar) square

On simplifying we get,
Lambda is equal to [summation (xi minus x bar) the whole square plus summation (yi minus y bar) the whole square divided by summation (xi minus mu cap)the whole square plus summation (yi minus mu cap)the whole square]the whole to the power 'm' plus 'n' divided by two.

Consider summation (xi minus mu cap) the whole square is equal to summation (xi minus x bar) the whole square plus 'n' into (x bar minus mu cap) the whole square.
'n' into (x bar minus mu cap) the whole square is equal to 'n' into [x bar minus 'n' x bar plus 'm' 'y' bar by 'm' plus 'n'] the whole square equal to 'n' 'm' square into (x bar minus y bar) the whole square by ('m' plus 'n')the whole square.
Summation (xi minus mu cap) the whole square is equal to summation (xi minus x bar) the whole square plus 'n' 'm' square into (x bar minus y bar) the whole square by (m plus n) the whole square.

3. Application 1 Contd

Similarly,
Summation (yi minus mu cap) the whole square is equal to summation (yi minus y bar) the whole square plus ‘m’ ‘n’ square into (x bar minus y bar) the whole square by (m plus n) the whole square.
Summation (xi minus mu cap) the whole square plus summation (yi minus mu cap) the whole square is equal to summation (xi minus x bar) the whole square plus summation (yi minus y bar) the whole square plus ‘n’ ‘m’ into (x bar minus y bar) the whole square by (m plus n).
Lambda is equal to [summation (xi minus x bar) the whole square plus summation (yi minus y bar) the whole square by summation (xi minus x bar) the whole square plus summation (yi minus y bar)the whole square plus ‘n’ ‘m’ into (x bar minus y bar) the whole square by (m plus n)] to the power ‘m’ plus ‘n’ by two.

Lambda is equal to one by one plus ‘n’ ‘m’ into (x bar minus y bar)the whole square by one plus (m plus n) by summation (xi minus x bar) the whole square plus summation (yi minus y bar) the whole square, the whole to the power ‘m’ plus ‘n’ by two
Which is equal to one by one plus‘t’ square by (m plus n minus two) to the power ‘m’ plus ‘n’ by two.
Where ‘t’ is equal to (x bar minus y bar) by square root of (one by n plus one by m) divided by square root of summation (xi minus x bar) the whole square plus summation (yi minus y bar) the whole square by ‘m’ plus ‘n’ minus two.

Now lambda less than or equal to lambda alpha implies one by one plus ‘t’ square by (m plus n minus two) the whole to the power ‘m’ plus ‘n’ by two less than or equal to lambda alpha.
One plus ‘t’ square by (m plus n minus two) is greater than or equal to lambda alpha to the power minus two by ‘m’ plus ‘n’
‘t’ square greater than or equal to lambda alpha to the power minus two by (m plus n minus one) into (m plus n minus two) which implies modulus of ‘t’ is greater than or equal to lambda one.

Now size of the test is equal to alpha implies Probability of reject H naught when H naught is true is equal to alpha.
This implies probability of modulus of‘ ‘t’ greater than or equal to lambda one is equal to alpha.
When mu one is equal to mu two is equal to mu, t is distributed as student’s t variable with (m plus n minus two) degrees of freedom.
From the table of probabilities of Students ‘t’ distribution we can read lambda one as ‘t’ alpha into (m plus n minus two)

A practical procedure for testing the hypothesis here is as follows:
To reject H naught if modulus of (x bar minus y bar) divided by square root of (one by ‘n’ plus one by ‘m’) the whole divided by summation (xi minus x bar) the whole square plus summation (yi minus y bar) the whole square by ‘m’ plus ‘n’ minus two is greater than ‘t’ alpha into (m plus n minus two)

One sided tests

In case we have to test H naught: mu one is equal to mu two against the alternative hypothesis H one: mu one greater is than mu two.
Then we obtain the following test procedure.
To reject H naught if the computed value of 't' exceeds the table value of 't' two alpha into (m plus n minus two).

Similarly, in case of testing H naught: mu one is equal to mu two against the alternative hypothesis Hone: mu one is less than mu two, then we obtain the following test procedure.
To reject H naught if the computed value of 't' is less than the table value minus 't' into two alpha into (m plus n minus two)

4. Application 2

Application two
To test the null hypothesis H naught: mu one is equal to mu two against the alternative hypothesis H one: mu one not is equal to mu two where mu one and mu two are the means of two Normal populations with a known common variance sigma square.

Let x one, x two, etc till x n be a random sample of size 'n' from a Normal population with mean mu one and let y one, y two, etc till 'y' 'm' be a random sample of size 'm' from a Normal population with mean mu two.
Then the likelihood ratio lambda is given by:
lambda is equal to Supremum of (one by sigma into square root two pi) whole to the power 'n' into 'e' to the power minus one by two sigma square into summation (xi minus mu one) the whole square into (one by sigma into square root of two pi) to the power 'm' into 'e' to the power minus one by two sigma square into summation (yi minus mu two) the whole square when mu one is equal to mu two is equal to mu.
Divided by Supremum of (one by sigma into square root of two pi) to the power 'n' into 'e' to the power minus one by two sigma square into summation (xi minus mu one) the whole square into (one by sigma into square root of two pi) to the power 'm' into 'e' to the power minus one by two sigma square into summation (yi minus mu two) the whole square under mu one coma mu two.

Under the null hypothesis the m.l.e's of mu one and mu two are obtained as mu cap is equal to 'n' 'x' bar plus 'm' 'y' bar by (m plus n).
Otherwise the m. l. e's are mu one is equal to 'x' bar and mu two cap is equal to 'y' bar.

Thus substituting the m.l.e's appropriately we have
Lambda is equal to 'e' to the power minus one by two sigma square into [summation (xi minus mu cap) the whole square plus summation (yi minus mu cap)the whole square]
Divided by 'e' to the power minus one by two sigma square into [summation (xi minus x bar)the whole square plus summation (yi minus y bar)the whole square]
Which is equal to 'e' to the power minus one by two sigma square into [summation (xi minus mu cap)the whole square plus summation (yi minus mu cap) the whole square] minus summation (xi minus x bar)the whole square minus summation (yi minus y bar) square]

On simplifying as in application one we get: summation (xi minus mu cap) the whole square plus summation (yi minus mu cap) the whole square is equal to summation (xi minus x bar) the whole square plus summation (yi minus y bar) the whole square plus 'n' 'm' into (x bar minus y bar) the whole square divided by (m plus n).

Lambda less than or equal to lambda alpha implies 'e' to the power minus one by two sigma square into ['n' into 'm' into (x bar minus y bar) the whole square by (m plus n)] less than or equal to lambda alpha.
Implies [x bar minus y bar by sigma by square root of (one by 'm' plus one by 'n') the whole square is greater than minus two into Ln lambda alpha.
Which implies modulus of [x bar minus y bar by sigma by square root of (one by 'm' plus one by 'n') is greater than square root of minus two Ln lambda alpha which is equal to

lambda one, say.

Now size of the test is equal to alpha implies Probability of reject H naught when H naught is true is equal to alpha.
Which implies probability of lambda less than or equal to lambda alpha given mu one is equal to u two is equal to alpha.
This implies probability of modulus of [x bar minus y bar by sigma by square root of (one by 'm' plus one by 'n') is greater than lambda one is equal to alpha.

[X bar minus 'y' bar divided by sigma by square root of (one by 'm' plus one by 'n')] follows Normal distribution with mean zero and variance one, under Null hypothesis.
Which implies probability of modulus of Normal (zero coma one) is greater than lambda one] which is equal to alpha.
Now from the table of Normal Probabilities we can read lambda one equal to Z alpha divided by two. Using the relation among lambda alpha and lambda one, the test can be determined.

To test the null hypothesis H naught: mu one is equal to mu two against the alternative hypothesis H one: mu one is greater than mu two when the variances are known.
Here from the above , we obtain the following test procedure:
To reject the null hypothesis, if the computed value of Z exceeds the table value of Z alpha.

Similarly in case of testing the null hypothesis H naught mu one is equal to mu two against the alternative hypothesis H one: mu one is less than mu two, we obtain the following test procedure:
To reject the null hypothesis if the computed value of Z is less than the table value of minus Z alpha.

5. Case of Paired Observations

Case of paired observations (Dependent observations)
Let us now consider two sample of equal sizes and the two samples are not independent but the sample observations are paired together
For example suppose we want to test the efficacy of a particular drug, say for inducing sleep.

Let Xi and yi be the readings in the hours of sleep on the ith individual before and after the drug is given respectively. Here instead of applying the difference of the means test as discussed above we apply paired ‘T’ test as explained in the following slide.

Let (X i coma Y i) coma i be equal to one coma two etc till ‘n’ be ‘n’ pairs of observations with x and y denoting observations from the Normal population. We assume here that the observations in each pair are dependent on each other.

To test the Hypothesis H Naught: mu one is equal to mu two against the alternative hypothesis H one: mu one is not equal to mu two, where mu one and mu two are the means of two Normal populations. Let d be a new variable defined by ‘d i’ is equal to x i minus y i
The above testing of Hypothesis is same as H naught: mu ‘d’ is equal to zero against the alternative hypothesis H one: mu ‘d’ is not equal to zero.
Where mu ‘d’ is the mean of the variable ‘d’ and mu ‘d’ is equal to mu one minus mu two.
The above is a situation as that of application two of the last paper, that is, to test the mean of a Normal population with an unknown variance.

Therefore the test procedure is to reject H naught if modulus of ‘t’ is equal to modulus of ‘D’ bar by ‘S’ ‘D’ divided by square root of ‘n’ greater than or equal to t alpha into (n minus one).
Where ‘SD’ square is equal to summation (d i minus d bar) the whole square by (n minus 1)
Assume that (D i) follows Normal with mean mu one minus mu two and variance sigma ‘D’ square.
Then, ‘D’ bar follows Normal with mean zero and variance sigma ‘d’ square by ‘n’ under null hypothesis.

One sided tests:
Suppose we want to test the null hypothesis H naught : mu one is equal to mu two against the alternative hypothesis : mu one is greater than mu two, for correlated or dependent variables with unknown variance, the critical region is given by C is equal to ‘t’ greater than or equal to ‘t’ two alpha into (n minus one)

Suppose we want to test the null hypothesis H naught : mu one is equal to mu two against the alternative hypothesis : mu one is less than mu two, for correlated or dependent variables with unknown variance, the critical region is given by C is equal to ‘t’ less than minus ‘t’ two alpha into (n minus one)

Here’s a summary of our learning in this session where we have understood:

- LRTP for testing difference of means of two Normal populations when the variances are known and unknown
- LRTP for testing difference of means of two dependent samples