1. Introduction

Welcome to the series of E-learning modules on Likelihood Ratio Test Procedure for testing the mean and variance of Univariate Normal distribution. In this module we are going cover the Likelihood Ratio Test Procedure for testing mean of a Normal population when variance is known and unknown and also test criteria for testing variance of a Normal population when the mean is known and unknown.

By the end of this session, you will be able to:

- Apply Likelihood Ratio Test Procedure or LRTP to test the mean when the variance is known and unknown in case of a Normal population
- Apply Likelihood Ratio Test Procedure to test the variance of a Normal population when the mean is known and unknown

A test introduced by Neyman and Pearson for testing a hypothesis, simple or composite against a simple or composite alternative hypothesis is related to the maximum likelihood estimates.
Before deriving the Likelihood ratio test statistic for mean and variance of Univariate Normal population let us just recollect the Likelihood Ratio test procedure.

Suppose a composite null hypothesis, H naught: theta belongs to the parameter space under the null hypothesis H naught, is to be tested against a composite alternative hypothesis H one, theta belongs to parameter space under the alternative hypothesis H one.
Let a random sample x one, two, etc, x n of size n be drawn from the given population with probability density function f of (x, theta).

Let lambda be equal to Supremum of L of (theta , x one, x two, etc till x n) when theta belongs to the parameter space under the null hypothesis H naught divided by Supremum of L of (theta , x one, x two, etc till x n) when theta belongs to the entire parameter space omega.

The test procedure is as follows: If lambda is very large, that is, lambda is greater than or equal to lambda alpha, the null hypothesis is accepted.
If lambda is very small, that is, lambda is less than lambda alpha; the null hypothesis is to be rejected.
The constant lambda alpha is so chosen that the size of the test is alpha.

2. Application 1

Application:1
Now let us look into some applications of LRTP
Let x follow Normal (mu, sigma square) with a known variance sigma square. To test the null hypothesis H naught: mu equal to mu naught against the alternative hypothesis H one: mu naught equal to mu naught.

Let x one, x two…xn be a random sample of size n drawn from a Normal population with parameters mu and sigma square.
lambda is equal to Supremum of L of (mu , x one, x two, etc till x n) when mu is equal to mu naught divided by Supremum of L of (mu , x one, x two, etc till x n) when mu lies between minus infinity and infinity.
Where lambda alpha is such that the size of the test is equal to alpha.

lambda is equal to Supremum of (one by sigma root two pi) to the power n into 'e' to the power (minus one by two sigma square) into summation (xi minus mu) whole square when mu equal to mu naught.
Divided by, Supremum of (one by sigma root two pi) to the power 'n' into 'e' to the power (minus one by two sigma square) into summation (xi minus mu) the whole square when mu is between minus infinity and infinity.

The denominator attains the maximum value when the unknown parameter is substituted with its maximum likelihood estimate.
Thus by substituting mu with x bar, a sample mean in the denominator, we can make it attain its maximum value.

Lambda is equal to (one by sigma root two pi) to the power 'n' into 'e' to the power minus one by two sigma square into summation (xi minus mu naught) the whole square.
Divided by (one by sigma root two pi) to the power 'n' into 'e' to the power minus one by two sigma square into summation (xi minus x bar) the whole square.

This is equal to 'e' to the power minus one by two sigma square into [summation (xi minus mu naught) the whole square minus summation (xi minus x bar) the whole square].
which implies 'e' to the power minus 'n' by two sigma square into[mu naught square plus x bar square minus two x bar into mu naught]
Which implies 'e' to the power minus 'n' by two sigma square into[x bar minus mu naught] the whole square.

Now lambda is less than or equal to lambda alpha implies 'e' to the power minus 'n' by two sigma square into[x bar minus mu naught] the whole square is less than or equal to lambda alpha.
This implies minus n by two sigma square into[x bar minus mu naught] the whole square is less than or equal to Ln lambda alpha.

Which implies [x bar minus mu naught by sigma by root 'n'] the whole square greater than minus two Ln lambda alpha.

Which implies modulus of [x bar minus mu naught by sigma by root ‘n’] is greater than square root of minus two Ln lambda alpha is equal to lambda one (say).
Now size of the test is equal to alpha implies, probability of [lambda less than or equal to lambda alpha, given mu is equal to mu naught] is equal to alpha.
This implies, probability of modulus of [x bar minus mu naught by sigma by root ‘n’] is greater than lambda one is equal to alpha.

When xi follows Normal with parameters mu and sigma square a sample mean follows Normal with mean mu and variance sigma square by n.
Then [x bar minus mu naught by sigma by root n] follows Normal with mean zero and variance one under Null hypothesis.
Now from the table of Normal Probabilities we can read lambda one equal to Z alpha by two.
Using the relation between lambda alpha and lambda one equal to Z alpha by two, lambda alpha and the test can be determined.

3. One Sided Tests: Cases 1 and 2

One sided test case 1:
To test the null hypothesis H naught: mu is equal to mu naught against the alternative hypothesis H one: mu is greater than mu naught.
Here from the above we have, if mu naught is greater than the sample mean, estimate of mu is mu naught in the denominator then lambda equal to one.
In which case the null hypothesis should surely be accepted.

If mu naught is *less than* the sample mean, then the maximum likelihood estimate of mu is x bar.
Thus substituting mu with x bar, a sample mean in the denominator we can make it to attain its maximum value as above.

Then, we get, lambda is equal to (one by sigma root two pi) to the power 'n' into 'e' to the power minus one by two sigma square into summation (xi minus mu naught)the whole square.
Divided by (one by sigma root two pi) to the power 'n' into 'e' to the power minus one by two sigma square into summation (xi minus x bar) the whole square
Now, lambda less than or equal to lambda alpha implies [x bar minus mu naught by sigma by root n] the whole square is greater than minus two Ln lambda alpha.
Which implies [x bar minus mu naught by sigma by root n] is greater than square root of minus two Ln lambda alpha is equal to lambda one
Now size of the test is equal to alpha implies probability of [x bar minus mu naught by sigma by root n] is greater than lambda one is equal to alpha.

Now from the table of Normal Probabilities we can read lambda one equal to Z alpha.
Using the relation among lambda alpha, and lambda one equal to Z alpha, lambda alpha and hence the test can be determined.

Case 2:
To test the null hypothesis H naught: mu equal to mu naught against the alternative hypothesis H one: mu less than mu naught
Here from the above we have:
If mu naught is less than the sample mean, estimate of mu is mu naught in the denominator then lambda is equal to one, in which case the null hypothesis should surely be accepted.

If mu naught is greater than the sample mean, then the maximum likelihood estimate of mu is x bar.
Thus substituting mu with x bar, a sample mean in the denominator, we can make it to attain its maximum value as above.

Now we get, lambda is equal to (one by sigma into square root of two pi) to the power 'n' into 'e' to the power minus one by two sigma square into summation (xi minus mu naught)the whole square.
Divided by (one by sigma into root of two pi) to the power 'n' into 'e' to the power minus one by two sigma square into summation (xi minus x bar) the whole square.

Now lambda less than or equal to lambda alpha implies, [x bar minus mu naught divided by sigma by square root of n] the whole square is greater than minus two Ln lambda alpha.
Which implies [mu naught minus x bar by sigma by square root of n] is greater than square root of minus two Ln lambda alpha
Which implies [x bar minus mu naught by sigma by root n] is less than or equal to minus square root of minus two Ln lambda alpha is equal to minus lambda one.

Now size of the test is equal to alpha implies probability of [x bar minus mu naught by sigma by root n] is less than or equal to minus lambda one is equal to alpha.
Now from the table of Normal Probabilities we can read lambda one is equal to minus Z alpha.

Using the relation between lambda alpha, and lambda one equal to minus Z alpha, lambda alpha and hence the test can be determined.

4. Application 2

Let x follow Normal (mu, sigma square) with an unknown variance.
To test the null hypothesis H naught: mu is equal to mu naught against the alternative hypothesis H one: mu is not equal to mu naught.

lambda is equal to Supremum of L of (x one, x two, etc till x n) when mu is equal to mu naught , sigma square
Divided by Supremum of L of (x one, x two, etc till x n) under mu coma sigma square.
Where lambda alpha is such that the size of the test is equal to alpha.
Lambda is equal to Supremum of (one by sigma square) to the power ‘n’ by two into (one by square root of two pi) to the power ‘n’ into ‘e’ to the power minus one by two sigma square into summation (xi minus mu) the whole square under sigma square coma mu equal to mu naught.
Divided by Supremum of (one by sigma square) to the power ‘n’ by two into (one by square root of two pi) to the power ‘n’ into ‘e’ to the power minus one by two sigma square into summation (xi minus mu) whole square under sigma square coma mu.

Under the null hypothesis the m.l.e of sigma square is summation (xi minus mu naught) the whole square by ‘n’ otherwise the m.l.e’s of mu and sigma square are x bar and summation (xi minus x bar) the whole square by n.
In the expression of lambda by substituting the above and simplifying we get,
Lambda is equal to summation (xi minus x bar) the square by summation (xi minus mu naught) the whole square to the power ‘n’ by two.

Consider summation (xi minus mu naught) the whole square is equal to summation (xi minus x bar plus (x bar minus mu naught) the whole square which is equal to summation (xi minus x bar) the whole square plus ‘n’ into (x bar minus mu naught) the whole square, because summation (xi minus x bar) is equal to zero.

By substituting and simplifying we get lambda is equal to one by one plus ‘n’ into (x bar minus mu naught the whole square by summation (xi minus x bar) the whole square to the power ‘n’ by two.

Now lambda is less than or equal to lambda alpha implies modulus of ‘t’ greater than lambda one where ‘t’ is equal to x bar minus mu naught by ‘s’ by root ‘n’.

Now size of the test is equal to alpha implies Probability of modulus of ‘t’ is greater than lambda one is equal to alpha.
The quantity lambda one can be be read from the table of probabilities of Students ‘t’ distribution for (n minus one) degrees of freedom as lambda one is equal to ‘t’ alpha into ‘n’ minus one.

One sided tests: Case 1:
To test H naught: mu is equal to mu naught against the alternative hypothesis H one: mu is greater than mu naught, with unknown variance.
Lambda less than or equal to lambda alpha and size of the test equal to alpha implies

Probability of ‘t’ is greater than lambda one is equal to alpha.
From the table of Probabilities of ‘t’ distribution we can read lambda one as ‘t’ two alpha into ‘n’ minus one.

To test H naught: mu is equal to mu naught against the alternative H one: mu is less than mu naught, the variance unknown.
Lambda less than or equal to lambda alpha and size of the test is equal to alpha implies Probability of ‘t’ less than minus lambda one is equal to alpha.
From the table of Probabilities of ‘t’ distribution we can read lambda one as minus ‘t’ two alpha into ‘n’ minus one.
That is, we reject the null hypothesis if x bar minus mu naught by s by root n is less than minus ‘t’ two alpha (n minus one)

5. Applications 3 and 4

Application :3
To test the null hypothesis H naught: sigma is equal to sigma naught against the alternative, H one: sigma is not equal to sigma naught, for a known mean mu.
Lambda is equal to supremum of (one by sigma square) to the power 'n' by two into (one by square root of two pi) to the power 'n' into 'e' to the power minus one by two sigma square into summation (xi minus mu) the whole square when sigma is equal to sigma naught.
Divided by supremum of (one by sigma square) to the power 'n' by two into (one by square root of two pi) to the power 'n' into 'e' to the power minus one by two sigma square summation (xi minus mu) the whole square when sigma is greater than or equal to zero.

By substituting sigma square with summation (x i minus mu) the whole square by 'n' in the denominator we can make it to attain its maximum value and then on simplifying we get:
Lambda less than or equal to lambda alpha implies 'u' less than or equal to 'C' one or 'U' greater than or equal to 'C' two

Where 'U' is equal to summation (x i minus mu) the whole square by sigma naught square follows chi square with 'n' degrees of freedom.
Which implies probability of chi square less than or equal to 'C' one plus probability of chi square greater than or equal to 'C' two is equal to alpha
where 'C' one equal to chi square (one minus alpha by two) with 'n' degrees of freedom and 'C' two is equal to chi square (alpha by two) with 'n' degrees of freedom.

One sided tests:
To test the null hypothesis H naught: sigma is equal to sigma naught against the alternative H one: sigma is greater than sigma naught, for a known mean
We reject the null hypothesis if summation (x i minus mu) the whole square by sigma naught square exceeds chi square alpha with 'n' degrees of freedom.

To test the null hypothesis H naught: sigma is equal to sigma naught against the alternative H one: sigma is less than sigma naught, for a known mean
We reject the null hypothesis if summation (x i minus mu) the whole square by sigma naught square is less than chi square one minus alpha square with 'n' degrees of freedom.

Application :4
To test the null hypothesis H naught: sigma is equal to sigma naught against the null hypothesis H one: sigma is not equal to sigma naught, for an unknown mean.
This procedure is same as application three.
Under the null hypothesis the m.l.e of mu is equal to x bar.
Otherwise the m.l.e's of mu and sigma square are x bar and summation (x i minus x bar) the whole square by 'n'.

In the expression of lambda, by substituting the parameters with the m.l.e's and then simplifying we get,
Lambda less than or equal to lambda alpha implies 'U' less than or equal to 'C' one or 'U' greater than or equal to 'C' two where 'U' is equal to summation (x i minus x bar) the whole

square by sigma naught square follows chi square with ‘n’ minus one degrees of freedom.

Size of the test is equal to alpha which implies probability of chi square less than or equal to ‘C’ one plus probability of chi square greater than or equal to ‘C’ two is equal to alpha.
where ‘C’ one equal to chi square (one minus alpha by two) with (n minus one)degrees of freedom and ‘C’ two is equal to chi square (alpha by two) with (n minus one)degrees of freedom.

One sided tests:
To test the null hypothesis H naught: sigma is equal to sigma naught against H one: sigma is greater than sigma naught, for an unknown mean,
We reject the null hypothesis if summation (x i minus x bar) the whole square by sigma naught square exceeds chi square alpha with (n minus 1) degrees of freedom.

To test the null hypothesis H naught: sigma is equal to sigma naught against H one: sigma is less than sigma naught, for an unknown mean
We reject the null hypothesis if summation (x i minus x bar) the whole square by sigma naught square is less than chi square (one minus alpha) with (n minus one)degrees of freedom.

Here’s a summary of our learning in this session where we have understood the following:

- Likelihood Ratio Test Procedure or LRTP for testing mean of a Normal population when the variance is known
- LRTP for testing mean when the variance is unknown
- LRTP for testing variance of a Normal population when the variance is known and unknown