Frequently Asked Questions

1. Briefly explain a Likelihood ratio test procedure

Answer:

Suppose a composite null hypothesis $H_0 : \theta \in \Theta_0$ is to be tested against a composite alternative hypothesis $H_1 : \theta \in \Theta_1$

For testing the above null hypothesis a test procedure called Likelihood ratio test procedure is followed which is explained below.

Let a random sample x1,x2,…,xn of size n be drawn from the given population with p.d.f f(x,θ)

Let $\lambda = \dfrac{\underset{\theta \in \Theta_0}{Sup} \; L(\theta, x1, x2, ..., xn)}{\underset{\theta \in \Theta}{Sup} \; L(\theta, x1, x2, ..., xn)}$

Since the supremum in the denominator is over a larger set of numbers. $\lambda \le 1$. Also likelihood functions are nonnegative and hence $\lambda \ge 0$. Thus $0 \le \lambda \le 1$.

That is the test procedure is as follows:

If λ is very large that is $\lambda \ge \lambda_\alpha$ the null hypothesis is accepted and if λ is very small that is $\lambda < \lambda_\alpha$ the null hypothesis is to be rejected. The constant λ_α t is so chosen that the size of the test is α

2. In practical situation how do you test the average of a Normal population?

Answer:

To test the null hypothesis H0: μ=μ0 against the alternative hypothesis H1: μ≠ μ0 , when the variance is known $\left| \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \right|$ is computed and compared with the tabulated value $z_{\alpha/2}$

That is when the computed value of $\left| \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \right|$ exceeds $z_{\alpha/2}$ we reject the null hypothesis.

To test the null hypothesis $H_O : \mu = \mu_O$ against the alternative hypothesis $H_1 : \mu \ne \mu_O$ when the variance in unknown, $\left| \dfrac{\bar{x} - \mu_0}{s / \sqrt{n}} \right|$ is computed and if it exceeds the tabulated value of t_α (n-1) we reject the null hypothesis.

3. What is the test criterion to test the variance of a Normal population in practical cases?

Answer:

In practice, to test the null hypothesis H0: σ=σ0 against H1: σ ≠ σ0 , when the mean is known, $\dfrac{\sum_i (xi - \mu)^2}{{\sigma_0}^2}$ is computed and compared with the tabulated value. That is we reject the null hypothesis if $\dfrac{\sum_i (xi - \mu)^2}{{\sigma_0}^2} > {\chi_{\alpha/2}}^2(n)$ or $\dfrac{\sum_i (xi - \mu)^2}{{\sigma_0}^2} < {\chi_{1-\alpha/2}}^2(n)$

To test the null hypothesis H0: σ=σ0 against H1: σ≠σ0, when the mean is unknown $\frac{\sum_i (xi-\bar{x})^2}{\sigma_0^2}$ is computed and compared with the tabulated value. That is we reject the null hypothesis if $\frac{\sum_i (xi-\bar{x})^2}{\sigma_0^2} > \chi_{\alpha/2}{}^2(n-1)$ or $\frac{\sum_i (xi-\bar{x})^2}{\sigma_0^2} < \chi_{1-\alpha/2}{}^2(n-1)$

4. For a normal population with a known variance σ^2 derive LRTP to test the null hypothesis H0: μ=μ0 against the alternative hypothesis H1: μ≠ μ0

Answer:

Let $x_1, x_2 \ldots x_n$ be a random sample of size n from a Normal population with parameters μ and σ^2

$$\lambda = \frac{\underset{\mu\in\Theta_0}{Sup}\; L(\mu, x1, x2, ..., xn)}{\underset{\mu\in\Omega}{Sup}\; L(\mu, x1, x2, ..., xn)}$$

Where λ_α is such that

$$P[\lambda \le \lambda_\alpha / H_0] = \alpha ------(*)$$

$$\lambda = \frac{\underset{\mu=\mu_0}{Sup}\; L(\mu, x1, x2, ..., xn)}{\underset{-\infty<\mu<\infty}{Sup}\; L(\mu, x1, x2, ..., xn)}$$

$$\lambda = \frac{\underset{\mu=\mu_0}{Sup}\; (\frac{1}{\sigma\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}{\underset{-\infty<\mu<\infty}{Sup}\; (\frac{1}{\sigma\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu\;)^2}}$$

The denominator attains the maximum value when the unknown parameter is substituted by its maximum likelihood estimate. Thus substituting μ by $\bar{x}$, a sample mean in the denominator we can make it to attain its maximum value. Therefore

$$\lambda = \frac{(\frac{1}{\sigma\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu_0)^2}}{(\frac{1}{\sigma\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\bar{x})^2}}$$

$$= e^{\frac{-1}{2\sigma^2}[\sum_i (xi-\mu_0)^2 - \sum_i (xi-\bar{x})^2]}$$

$$\Rightarrow e^{\frac{-1}{2\sigma^2}[\sum_i xi^2 + n\mu_0^2 - 2\mu_0 \sum_i xi \quad -(\sum_i xi^2 + n\bar{x}^2 - 2\bar{x}\sum_i xi \quad)]}$$

$$\Rightarrow e^{\frac{-1}{2\sigma^2}[n\mu_0^2 + n\bar{x}^2 - 2n\bar{x}\mu_0]}$$

$$\Rightarrow e^{\frac{-n}{2\sigma^2}[\mu_0^2 + \bar{x}^2 - 2\bar{x}\mu_0]} \Longrightarrow e^{\frac{-n}{2\sigma^2}[\bar{x}-\mu_0]^2}$$

Now

$$\lambda \le \lambda_\alpha \Rightarrow e^{\frac{-n}{2\sigma^2}[\bar{x}-\mu_0]^2} \le \lambda_\alpha$$

$$\Rightarrow \frac{-n}{2\sigma^2}[\bar{x}-\mu_0]^2 \le \ln \lambda_\alpha$$

$$\Rightarrow \left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)^2 \left(\frac{-1}{2}\right) \le \ln \lambda_\alpha$$

$$\Rightarrow \left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)^2 > -2\ln \lambda_\alpha$$

$$\Rightarrow \left|\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right| > \sqrt{-2\ln \lambda_\alpha} = \lambda_1 \text{ (say)}$$

Now size of the test =α $\Rightarrow P[\text{Re } ject H_0 / H_0 true] = \alpha$

$$\Rightarrow P[\lambda \le \lambda_\alpha / \mu = \mu_0] = \alpha$$

$$\Rightarrow P[\left|\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right| > \lambda_1] = \alpha$$

When xi follows Normal with parameters μ and σ^2, a sample mean $\bar{x} \sim N(\mu, \sigma^2/n)$. Then $\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$ under Null hypothesis

$$\Rightarrow P[|N(0,1)| > \lambda_1] = \alpha$$

Now from the table of Normal Probabilities we can read $\lambda_1 = Z_{\alpha/2}$. Using the relation among λ_α, and $\lambda_1 = Z_{\alpha/2}$, λ_α and hence the test can be determined.

5. Deduce LRTP to test the null hypothesis H0: μ=μ0 against the alternative hypothesis H1: μ > μ0 when the variance is given

Answer:

Let x_1, $x_2 \ldots x_n$ be a random sample of size n from a Normal population with parameters μ and σ^2

$$\lambda = \frac{\underset{\mu\in\Theta_0}{Sup} L(\mu, x1, x2, ..., xn)}{\underset{\mu\in\Omega}{Sup} L(\mu, x1, x2, ..., xn)}$$

$$P[\lambda \le \lambda_\alpha / H_0] = \alpha - - - - - (*)$$

Where λα is such that

$$\lambda = \frac{\underset{\mu=\mu_0}{Sup}\ L(\mu, x1, x2, ..., xn)}{\underset{-\infty<\mu<\infty}{Sup}\ L(\mu, x1, x2, ..., xn)}$$

$$\lambda = \frac{\underset{\mu=\mu_0}{Sup}\ (\frac{1}{\sigma\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}{\underset{-\infty<\mu<\infty}{Sup}\ (\frac{1}{\sigma\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu\ \ )^2}}$$

If μ_0 is greater than the sample mean, estimate of μ is μ_0 in the denominator then λ=1, in which case the null hypothesis should surely be accepted.

If μ_0 is less than the sample mean then the maximum likelihood estimate of μ is ., Thus substituting μ by $\bar{x}$, a sample mean in the denominator we can make it to attain its maximum value as above we get

$$\lambda = \frac{(\frac{1}{\sigma\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu_0)^2}}{(\frac{1}{\sigma\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\bar{x})^2}}$$

Now

$$\lambda \le \lambda_\alpha \Rightarrow$$

$$\Rightarrow \left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)^2 > -2\ln\lambda_\alpha$$

$$\Rightarrow \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} > \sqrt{-2\ln\lambda_\alpha} = \lambda_1 \text{ (say)}$$

Now size of the test =α $\Rightarrow P[\text{Re}\ jectH_0 / H_0 true] = \alpha$

$$\Rightarrow P[\lambda \le \lambda_\alpha / \mu = \mu_0] = \alpha$$

$$\Rightarrow P[\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} > \lambda_1] = \alpha$$

$$\Rightarrow P[N(0,1) > \lambda_1] = \alpha$$

Now from the table of Normal Probabilities we can read λ1 = Zα. Using the relation among λα, and λ1 = Zα, λα and hence the test can be determined. For our practical purposes to test the null hypothesis H0: μ=μ0 against the alternative hypothesis H1: μ>μ0 , $\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$ is computed and compared with the tabulated value z_α. That is we reject the null hypothesis if $\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$ exceeds Z_α

6. To test the null hypothesis H0: μ=μ0 against the alternative hypothesis H1: μ< μ0 , derive LRTP when variance of the population is known

Answer:

Let x_1, x_2…xn be a random sample of size n from a Normal population with parameters μ and σ^2

$$\lambda = \frac{\underset{\mu\in\Theta_0}{Sup}\ L(\mu, x1, x2, ..., xn)}{\underset{\mu\in\Omega}{Sup}\ L(\mu, x1, x2, ..., xn)}$$

$$P[\lambda \le \lambda_\alpha / H_0] = \alpha -----(*)$$

Where λα is such that

$$\lambda = \frac{\underset{\mu=\mu_0}{Sup}\ L(\mu, x1, x2, ..., xn)}{\underset{-\infty<\mu<\infty}{Sup}\ L(\mu, x1, x2, ..., xn)}$$

$$\lambda = \frac{\underset{\mu=\mu_0}{Sup}\ (\frac{1}{\sigma\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}{\underset{-\infty<\mu<\infty}{Sup}\ (\frac{1}{\sigma\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu\ )^2}}$$

$$\lambda = \frac{\underset{\mu=\mu_0}{Sup}\ (\frac{1}{\sigma\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}{\underset{\mu\le\mu_0}{Sup}\ (\frac{1}{\sigma\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu\ )^2}}$$

If μ_0 is less than the sample mean, estimate of μ is μ_0 in the denominator then λ=1, in which case the null hypothesis should surely be accepted.

If μ_0 is greater than the sample mean then the maximum likelihood estimate of μ is ., Thus substituting μ by $\bar{x}$, a sample mean in the denominator we can make it to attain its maximum value as above we get

$$\lambda = \frac{(\frac{1}{\sigma\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu_0)^2}}{(\frac{1}{\sigma\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\bar{x})^2}}$$

Now

$$\lambda \le \lambda_\alpha \Rightarrow$$

$$\Rightarrow \left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)^2 > -2\ln\lambda_\alpha$$

$$\Rightarrow \frac{\mu_0-\bar{x}}{\sigma/\sqrt{n}} > \sqrt{-2\ln\lambda_\alpha} = \lambda_1 \text{ (say)}$$

$$\Rightarrow \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \le -\sqrt{-2\ln\lambda_\alpha} = -\lambda_1$$

Now size of the test =α $\Rightarrow P[\text{Re}\,jectH_0 / H_0 true] = \alpha$

$$\Rightarrow P[\lambda \le \lambda_\alpha / \mu = \mu_0] = \alpha$$

$$\Rightarrow P[\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \le -\lambda_{11}] = \alpha$$

$$\Rightarrow P[N(0,1) \le -\lambda_1] = \alpha$$

Using the relation among λ_α, and $\lambda_1 = Z_\alpha$, λ_α and hence the test can be determined. For our practical purposes to test the null hypothesis H0: μ=μ0 against the alternative hypothesis H1: μ<μ0 , $\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$ is computed and compared with the tabulated value $-Z_\alpha$. That is we reject the null hypothesis if $\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$ is less than - Z_α

7. Let $x \sim N(\mu,\sigma^2)$ Obtain LRTP to test the null hypothesis H0: μ=μ0 against the alternative hypothesis H1: μ≠ μ0 , with an unknown variance σ^2

Answer:

Let $x_1, x_2 \ldots x_n$ be a random sample of size n from a Normal population with parameters μ and σ^2

$$\lambda = \frac{\underset{\sigma^2,\mu=\mu_0}{Sup} \; L(x1, x2, ..., xn)}{\underset{\sigma^2,\mu}{Sup} \; L(x1, x2, ..., xn)}$$

$$P[\lambda \le \lambda_\alpha / H_0] = \alpha - - - - - (*)$$

Where λα is such that

$$\lambda = \frac{\underset{\sigma^2, \mu=\mu_0}{Sup} \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}{\underset{\sigma^2, \mu}{Sup} \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu\ )^2}}$$

Under the null hypothesis the m.l.e of σ^2 is $\frac{\sum(xi-\mu_0)^2}{n}$ otherwise the m.l.e's of μ and σ^2 are $\bar{x}$ and $\frac{\sum(xi-\bar{x})^2}{n}$. In the expression of λ substituting the parameters by m.l.e's so that numerator and denominator attains its supremum we have

$$\lambda = \frac{\left(\frac{1}{\frac{\sum(xi-\mu_0)^2}{n}}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\frac{\frac{-1}{\sum(xi-\mu_0)^2}\sum_i (xi-\mu_0)^2}{2\frac{}{n}}}}{\left(\frac{1}{\frac{\sum(xi-\bar{x})^2}{n}}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\frac{\frac{-1}{\sum(xi-\bar{x})^2}\sum_i (xi-\bar{x})^2}{2\frac{}{n}}}}$$

$$= \left(\frac{\sum(xi-\bar{x})^2}{\sum(xi-\mu_0)^2}\right)^{\frac{n}{2}}$$

Consider

$\sum(xi-\mu_0)^2 =$

$$\sum[(xi-\bar{x})+(\bar{x}-\mu_0)]^2 = \sum(xi-\bar{x})^2 + n(\bar{x}-\mu_0)^2 because \sum(xi-\bar{x}) = 0$$

$$\lambda = \left(\frac{\sum (xi - \bar{x})^2}{\sum (xi - \bar{x})^2 + n(\bar{x} - \mu_0)^2} \right)^{\frac{n}{2}} == \left(\frac{1}{1 + \frac{n(\bar{x} - \mu_0)^2}{\sum (xi - \bar{x})^2}} \right)^{\frac{n}{2}}$$

Now

$$\lambda \le \lambda_\alpha \Rightarrow \left(\frac{1}{1 + \frac{n(\bar{x} - \mu_0)^2}{\sum (xi - \bar{x})^2}} \right)^{\frac{n}{2}} \le \lambda_\alpha$$

$$\Rightarrow \left(\frac{1}{1 + \frac{t^2}{n-1}} \right)^{\frac{n}{2}} \le \lambda_\alpha \Rightarrow \left(\frac{1}{1 + \frac{t^2}{n-1}} \right) \le \lambda_\alpha^{\frac{2}{n}}$$

Where t= $\left(\frac{\bar{x} - \mu_0}{s/\sqrt{n}} \right)$ and $s^2 = \frac{\sum (xi - \bar{x})^2}{n-1}$ then $\frac{t^2}{n-1} = \frac{n(\bar{x} - \mu_0)^2}{\sum (xi - \bar{x})^2}$

$$\Rightarrow \left(1 + \frac{t^2}{n-1} \right) > \lambda_\alpha^{\frac{-2}{n}} \Rightarrow t^2 \ge (n-1)(\lambda_\alpha^{\frac{-2}{n}} - 1)$$

$$\Rightarrow |t| \ge \sqrt{(n-1)(\lambda_\alpha^{\frac{-2}{n}} - 1)} = \lambda_1 (say)$$

Now size of the test =α $\Rightarrow P[\text{Re}\, ject H_0 / H_0 true] = \alpha$

$$\Rightarrow P[\lambda \le \lambda_\alpha / \mu = \mu_0] = \alpha$$

$$\Rightarrow P[|t| > \lambda_1] = \alpha$$

The quantity λ_1 can be read from the table of probabilities if Students t distribution for (n-1) degrees of freedom as $\lambda_1 = t_\alpha(n-1)$. Using the relation among λ_1 and λ_α , the test can be determined.

8. To test the null hypothesis H0: μ=μ0 against the alternative hypothesis H1: μ> μ0 , the variance unknown obtain LRTP.
Answer:

Let x_1, $x_2 \ldots x_n$ be a random sample of size n from a Normal population with parameters μ and σ^2

$$\lambda = \frac{\underset{\sigma^2, \mu=\mu_0}{Sup} \quad L(x1, x2, \ldots., xn)}{\underset{\sigma^2, \mu}{Sup} \quad L(x1, x2, \ldots., xn)}$$

$$P[\lambda \le \lambda_\alpha / H_0] = \alpha -----(*)$$

Where λα is such that

$$\lambda = \frac{\underset{\sigma^2, \mu=\mu_0}{Sup} \quad (\frac{1}{\sigma^2})^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}{\underset{\sigma^2, \mu}{Sup} \quad (\frac{1}{\sigma^2})^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}$$

Under the null hypothesis the m.l.e of σ^2 is $\frac{\sum (xi-\mu_0)^2}{n}$ otherwise the m.l.e's of μ and σ^2 are $\bar{x}$ and $\frac{\sum (xi-\bar{x})^2}{n}$. In the expression of λ substituting the parameters by m.l.e's so that numerator and denominator attains its supremum we have

$$\lambda = \frac{\left(\frac{1}{\frac{\sum (xi-\mu_0)^2}{n}}\right)^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\frac{\sum (xi-\mu_0)^2}{n}}\sum_i (xi-\mu_0)^2}}{\left(\frac{1}{\frac{\sum (xi-\bar{x})^2}{n}}\right)^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\frac{\sum (xi-\bar{x})^2}{n}}\sum_i (xi-\bar{x})^2}}$$

$$= \left(\frac{\sum (xi-\bar{x})^2}{\sum (xi-\mu_0)^2}\right)^{\frac{n}{2}}$$

Consider

$$\sum(xi-\mu_0)^2 =$$

$$\sum[(xi-\bar{x})+(\bar{x}-\mu_0)]^2 = \sum(xi-\bar{x})^2 + n(\bar{x}-\mu_0)^2 because \sum(xi-\bar{x}) = 0$$

$$\lambda = \left(\frac{\sum(xi-\bar{x})^2}{\sum(xi-\bar{x})^2 + n(\bar{x}-\mu_0)^2}\right)^{\frac{n}{2}} == \left(\frac{1}{1+\frac{n(\bar{x}-\mu_0)^2}{\sum(xi-\bar{x})^2}}\right)^{\frac{n}{2}}$$

$$\lambda \le \lambda_\alpha \Rightarrow \frac{\bar{x}-\mu_0}{s/\sqrt{n}} > \lambda_1$$

Now size of the test =α $\Rightarrow P[\text{Re}\, jectH_0 / H_0 true] = \alpha$

$$\Rightarrow P[\lambda \le \lambda_\alpha / \mu = \mu_0] = \alpha$$

$$\Rightarrow P[\frac{\bar{x}-\mu_0}{s/\sqrt{n}} > \lambda_1] = \alpha$$

$$\Rightarrow P[t > \lambda_1] = \alpha$$

Now from the table of probabilities of t distribution we can read $\lambda_1 = t_{2\alpha}(n-1)$ For our practical purposes to test the null hypothesis H0: μ=μ0 against the alternative hypothesis H1: μ>μ0 , $\frac{\bar{x}-\mu_0}{s/\sqrt{n}}$ is computed and compared with the tabulated value $t_{2\alpha}(n-1)$

That is we reject the null hypothesis if $\frac{\bar{x}-\mu_0}{s/\sqrt{n}}$ exceeds $t_{2\alpha}(n-1)$

9. Obtain a test procedure to test the null hypothesis H_0: $\mu=\mu_0$ against the alternative hypothesis H_1: $\mu< \mu_0$ for an unknown variance.

Answer:

Let x_1, x_2…xn be a random sample of size n from a Normal population with parameters μ and σ^2

$$\lambda = \frac{\underset{\sigma^2,\mu=\mu_0}{Sup} \quad L(x1, x2, ..., xn)}{\underset{\sigma^2,\mu}{Sup} \; L(x1, x2, ..., xn)}$$

$$P[\lambda \le \lambda_\alpha / H_0] = \alpha -----(*)$$

Where λ_α is such that

$$\lambda = \frac{\underset{\sigma^2, \mu=\mu_0}{Sup} \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}{\underset{\sigma^2, \mu}{Sup} \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu\ )^2}}$$

Under the null hypothesis the m.l.e of σ^2 is $\frac{\sum(xi-\mu_0)^2}{n}$ otherwise the m.l.e's of μ and σ^2 are $\bar{x}$ and $\frac{\sum(xi-\bar{x})^2}{n}$. In the expression of λ substituting the parameters by m.l.e's so that numerator and denominator attains its supremum we have

$$\lambda = \frac{\left(\frac{1}{\frac{\sum(xi-\mu_0)^2}{n}}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\frac{\frac{-1}{\sum(xi-\mu_0)^2}\sum_i (xi-\mu_0)^2}{2\,n}}}{\left(\frac{1}{\frac{\sum(xi-\bar{x})^2}{n}}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\frac{\frac{-1}{\sum(xi-\bar{x})^2}\sum_i (xi-\bar{x})^2}{2\,n}}}$$

$$= \left(\frac{\sum(xi-\bar{x})^2}{\sum(xi-\mu_0)^2}\right)^{\frac{n}{2}}$$

Consider

$\sum(xi-\mu_0)^2 =$

$$\sum[(xi-\bar{x})+(\bar{x}-\mu_0)]^2 = \sum(xi-\bar{x})^2 + n(\bar{x}-\mu_0)^2 because \sum(xi-\bar{x}) = 0$$

$$\lambda = \left(\frac{\sum (xi-\bar{x})^2}{\sum (xi-\bar{x})^2 + n(\bar{x}-\mu_0)^2} \right)^{\frac{n}{2}} == \left(\frac{1}{1+\frac{n(\bar{x}-\mu_0)^2}{\sum (xi-\bar{x})^2}} \right)^{\frac{n}{2}}$$ To test the null

Now

$$\lambda \le \lambda_\alpha \Rightarrow \frac{\bar{x}-\mu_0}{s/\sqrt{n}} \le -\lambda_1$$

Now size of the test =α

$$\Rightarrow P[\lambda \le \lambda_\alpha / \mu = \mu_0] = \alpha$$

$$\Rightarrow P[\frac{\bar{x}-\mu_0}{s/\sqrt{n}} \le -\lambda_{11}] = \alpha$$

For our practical purposes to test the null hypothesis

H0: μ=μ0 against the alternative hypothesis H1: μ<μ0, $\frac{\bar{x}-\mu_0}{s/\sqrt{n}}$ is computed and compared with the tabulated value $t_{2\alpha}(n-1)$

That is we reject the null hypothesis if $\frac{\bar{x}-\mu_0}{s/\sqrt{n}}$ is less than $-t_{2\alpha}(n-1)$

10. Let $x \sim N(\mu, \sigma^2)$ with a known mean μ · Derive LRTP to test the null hypothesis H0: $\sigma=\sigma_0$ against H1: $\sigma \ne \sigma_0$

Answer:

Let $x_1, x_2 \ldots, x_n$ be a random sample of size n from a Normal population with parameters μ and σ^2

$$\lambda = \frac{\underset{\sigma=\sigma_0}{Sup}\ L(x1, x2, \ldots, xn)}{\underset{\sigma \ge 0}{Sup}\ L(x1, x2, \ldots, xn)}$$

$$P[\lambda \le \lambda_\alpha / H_0] = \alpha -----(*)$$

Where λα is such that

$$\lambda = \frac{\underset{\sigma\ =\sigma_0}{Sup}\ (\frac{1}{\sigma^2})^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}{\underset{\sigma\ \ge 0}{Sup}\ (\frac{1}{\sigma^2})^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu\ )^2}}$$

The denominator attains the maximum value when the unknown parameter is substituted by its maximum likelihood estimate. Thus substituting $\sigma^2 = \dfrac{\sum(xi-\mu_0)^2}{n}$ in the denominator we can make it to attain its maximum value. Therefore

$$\lambda = \frac{(\frac{1}{\sigma_0{}^2})^{\frac{n}{2}}(\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma_0{}^2}\sum_i (xi-\mu)^2}}{\left(\frac{1}{\frac{\sum(xi-\mu)^2}{n}}\right)^{\frac{n}{2}}(\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\frac{\sum(xi-\mu)^2}{n}}\sum_i (xi-\mu)^2}}$$

$$\Rightarrow \left(\frac{\sum(xi-\mu)^2}{\sigma_0^2}\right)^{\frac{n}{2}} n^{\frac{-n}{2}} e^{\frac{n}{2}} e^{(\frac{-1}{2})\frac{\sum(xi-\mu)^2}{\sigma_0^2}}$$

$$\Rightarrow e^{\frac{-u}{2}} u^{\frac{n}{2}} k \text{ where } u = \frac{\sum(xi-\mu)^2}{\sigma_0^2}, k = n^{\frac{-n}{2}} e^{\frac{n}{2}}$$

$$\lambda \le \lambda_\alpha \Rightarrow e^{\frac{-u}{2}} u^{\frac{n}{2}} k \le \lambda_\alpha \Rightarrow e^{\frac{-u}{2}} u^{\frac{n}{2}} \le \frac{\lambda_\alpha}{k}$$

Now

$$\Rightarrow u \le c_1 \text{ or } u \ge c_2 \text{ where } e^{\frac{-c_1}{2}} c_1^{\frac{n}{2}} = e^{\frac{-c_2}{2}} c_2^{\frac{n}{2}} = \frac{\lambda_\alpha}{k}$$

Now size of the test =α $\Rightarrow P[\text{Re}\, ject H_0 / H_0 true] = \alpha$

$$\Rightarrow P[\lambda \le \lambda_\alpha / \sigma = \sigma_0] = \alpha$$

$$\Rightarrow P[u \le c_1 or u \ge c_2 / \sigma = \sigma_0] = \alpha$$

$$\Rightarrow P[\chi^2(n) \le c_1] + P[\chi^2(n) \ge c_2] = \alpha$$

When xi follows Normal with parameters μ and σ^2, and when $\sigma=\sigma_0$, $\frac{\sum_i (xi-\mu)^2}{\sigma_0^2} \sim \chi_{\alpha/2}^2(n)$. Therefore we find the constants C_1 and C_2

such that $e^{\frac{-c_1}{2}} c_1^{\frac{n}{2}} = e^{\frac{-c_2}{2}} c_2^{\frac{n}{2}} = \frac{\lambda_\alpha}{k}$

And $P[\chi^2(n) \le c_1] + P[\chi^2(n) \ge c_2] = \alpha$ where $c_1 = \chi_{1-\alpha/2}^2(n)$ and $C_2 = \chi_{\alpha/2}^2(n)$

11. Derive LRTP to test the null hypothesis H_0: $\sigma=\sigma_0$ against H_1: $\sigma>\sigma_0$ when the mean of a Normal population is known.

Answer:

Let $x_1, x_2 \ldots x_n$ be a random sample of size n from a Normal population with parameters μ and σ^2

$$\lambda = \frac{\underset{\sigma=\sigma_0}{Sup}\; L(x1, x2, \ldots, xn)}{\underset{\sigma\ge 0}{Sup}\; L(x1, x2, \ldots, xn)}$$

$$P[\lambda \le \lambda_\alpha / H_0] = \alpha ------(*)$$

Where λα is such that

$$\lambda = \frac{\underset{\sigma=\sigma_0}{Sup}\; (\frac{1}{\sigma^2})^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}{\underset{\sigma\ge 0}{Sup}\; (\frac{1}{\sigma^2})^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}$$

The denominator attains the maximum value when the unknown parameter is substituted by its maximum likelihood estimate. Thus substituting $\sigma^2 = \frac{\sum (xi-\mu_0)^2}{n}$ in the denominator we can make it to attain its maximum value. Therefore

$$\lambda = \frac{(\frac{1}{\sigma_0^2})^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma_0^2}\sum_i (xi-\mu)^2}}{\left(\frac{1}{\frac{\sum (xi-\mu)^2}{n}}\right)^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\frac{\sum (xi-\mu)^2}{n}}\sum_i (xi-\mu)^2}}$$

$$\Rightarrow \left(\frac{\sum (xi-\mu)^2}{\sigma_0^2} \right)^{\frac{n}{2}} n^{\frac{-n}{2}} e^{\frac{n}{2}} e^{(\frac{-1}{2})\frac{\sum (xi-\mu)^2}{\sigma_0^2}}$$

$$\Rightarrow e^{\frac{-u}{2}} u^{\frac{n}{2}} k \text{ where } u = \frac{\sum (xi-\mu)^2}{\sigma_0^2}, k = n^{\frac{-n}{2}} e^{\frac{n}{2}}$$

$$\lambda \le \lambda_\alpha \Rightarrow \frac{\sum_i (xi-\mu)^2}{\sigma_0{}^2} > C_1 = \chi_\alpha{}^2(n)$$

Now size of the test =α $\Rightarrow P[\text{Re } jectH_0 / H_0 true] = \alpha$

$$\Rightarrow P[\lambda \le \lambda_\alpha / \sigma = \sigma_0] = \alpha$$

$$\Rightarrow P[\frac{\sum_i (xi-\mu)^2}{\sigma_0{}^2} > \chi_\alpha{}^2(n)] = \alpha$$

In practice to test the null hypothesis

H_0: $\sigma=\sigma_0$ against H_1: $\sigma>\sigma_0$, the mean is known $\frac{\sum_i (xi-\mu)^2}{\sigma_0{}^2}$ is computed and compared with the tabulated value $\chi_\alpha{}^2(n)$. That is we reject the null hypothesis if $\frac{\sum_i (xi-\mu)^2}{\sigma_0{}^2}$ exceeds $\chi_\alpha{}^2(n)$

12. Derive LRTP to test the null hypothesis H_0: $\sigma=\sigma_0$ against H_1: $\sigma<\sigma_0$ when the mean of a Normal population is known.

Answer:

Let $x_1, x_2 \ldots x_n$ be a random sample of size n from a Normal population with parameters μ and σ^2

$$\lambda = \frac{\underset{\sigma=\sigma_0}{Sup}\ L(x1, x2, ..., xn)}{\underset{\sigma \ge 0}{Sup}\ L(x1, x2, ..., xn)}$$

Where λα is such that

$$P[\lambda \le \lambda_\alpha / H_0] = \alpha -\!-\!-\!-\!-\!- (*)$$

$$\lambda = \frac{\underset{\sigma = \sigma_0}{Sup} \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}{\underset{\sigma \geq 0}{Sup} \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}$$

The denominator attains the maximum value when the unknown parameter is substituted by its maximum likelihood estimate. Thus substituting $\sigma^2 = \frac{\sum (xi-\mu_0)^2}{n}$ in the denominator we can make it to attain its maximum value. Therefore

$$\lambda = \frac{\left(\frac{1}{\sigma_0^2}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\frac{-1}{2\sigma_0^2}\sum_i (xi-\mu)^2}}{\left(\frac{1}{\frac{\sum (xi-\mu)^2}{n}}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\frac{-1}{2\frac{\sum (xi-\mu)^2}{n}}\sum_i (xi-\mu)^2}}$$

$$\Rightarrow \left(\frac{\sum (xi-\mu)^2}{\sigma_0^2}\right)^{\frac{n}{2}} n^{\frac{-n}{2}} e^{\frac{n}{2}} e^{\left(\frac{-1}{2}\right)\frac{\sum (xi-\mu)^2}{\sigma_0^2}}$$

$\Rightarrow e^{\frac{-u}{2}} u^{\frac{n}{2}} k$ where $u = \frac{\sum (xi-\mu)^2}{\sigma_0^2}, k = n^{\frac{-n}{2}} e^{\frac{n}{2}}$ How the assumptions of the model can be tested using LRTP

$$\lambda \leq \lambda_\alpha \Rightarrow \frac{\sum_i (xi-\mu)^2}{\sigma_0^2} < C_2 = \chi_{1-\alpha}^2(n)$$

Now size of the test =α $\Rightarrow P[\text{Re} \, jectH_0 / H_0 true] = \alpha$

$$\Rightarrow P[\lambda \leq \lambda_\alpha / \sigma = \sigma_0] = \alpha$$

$$\Rightarrow P\left[\frac{\sum_i (xi-\mu)^2}{\sigma_0^2} < \chi_{1-\alpha}^2(n)\right] = \alpha$$

In practice to test the null hypothesis

H0: $\sigma=\sigma_0$ against H1: $\sigma<\sigma_0$, the mean is known $\frac{\sum_i (xi-\mu)^2}{\sigma_0^2}$ is computed and compared with the tabulated value $\chi_{1-\alpha}^2(n)$. That is we reject the null hypothesis if $\frac{\sum_i (xi-\mu)^2}{\sigma_0^2}$ is less than $\chi_{1-\alpha}^2(n)$

13. Derive LRTP to test the null hypothesis H0: $\sigma=\sigma_0$ against H1: $\sigma\neq\sigma_0$, with an unknown mean μ

Answer:

Let $x_1, x_2 \ldots x_n$ be a random sample of size n from a Normal population with parameters μ and σ^2

$$\lambda = \frac{\underset{\mu,\sigma=\sigma_0}{Sup} \; L(x1, x2,...., xn)}{\underset{\mu,\sigma\geq 0}{Sup} \; L(x1, x2,...., xn)}$$

Where λα is such that

$$P[\lambda \leq \lambda_\alpha / H_0] = \alpha ------(*)$$

$$\lambda = \frac{\underset{\mu,\sigma \;\; =\sigma_0}{Sup} \;\; (\frac{1}{\sigma^2})^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}{\underset{\mu,\sigma \;\; \geq 0}{Sup} \;\; (\frac{1}{\sigma^2})^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu \;\;)^2}}$$

Under the null hypothesis the m.l.e of μ = $\overline{x}$ otherwise the m.l.e's of μ and σ^2 are $\overline{x}$ and $\frac{\sum (xi-\overline{x})^2}{n}$.

In the expression of λ substituting the parameters by m.l.e's so that numerator and denominator attains its supremum we have

$$\lambda = \frac{\left(\frac{1}{\sigma_0^2}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\frac{-1}{2\sigma_0^2}\sum_i (xi-\bar{x})^2}}{\left(\frac{1}{\frac{\sum (xi-\bar{x})^2}{n}}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\frac{-1}{2\frac{\sum (xi-\bar{x})^2}{n}}\sum_i (xi-\bar{x})^2}}$$

$$\Rightarrow \left(\frac{\sum (xi-\overline{x})^2}{\sigma_0^2}\right)^{\frac{n}{2}} n^{\frac{-n}{2}} e^{\frac{n}{2}} e^{\left(\frac{-1}{2}\right)\frac{\sum (xi-\overline{x})^2}{\sigma_0^2}}$$

$$\Rightarrow e^{\frac{-u}{2}} u^{\frac{n}{2}} k \text{ where } u = \frac{\sum (xi-\overline{x})^2}{\sigma_0^2}, k = n^{\frac{-n}{2}} e^{\frac{n}{2}}$$

$$\lambda \le \lambda_\alpha \Rightarrow e^{\frac{-u}{2}} u^{\frac{n}{2}} k \le \lambda_\alpha \Rightarrow e^{\frac{-u}{2}} u^{\frac{n}{2}} \le \frac{\lambda_\alpha}{k}$$

Now

$\Rightarrow u \le c_1$ or $u \ge c_2$ where $e^{\frac{-c_1}{2}} c_1^{\frac{n}{2}} = e^{\frac{-c_2}{2}} c_2^{\frac{n}{2}} = \frac{\lambda_\alpha}{k}$

Now size of the test =α $\Rightarrow P[\text{Reject}H_0 / H_0 true] = \alpha$

$$\Rightarrow P[\lambda \le \lambda_\alpha / \sigma = \sigma_0] = \alpha$$

$$\Rightarrow P[u \le c_1 or u \ge c_2 / \sigma = \sigma_0] = \alpha$$

$$\Rightarrow P[\chi^2(n) \le c_1] + P[\chi^2(n) \ge c_2] = \alpha$$

When xi follows Normal with parameters μ and σ^2 and when σ =

$$\frac{\sum_i (xi-\bar{x})^2}{\sigma_0^2} \sim \chi_{\alpha/2}^2 (n-1)$$

σ_0

Therefore we find the constants C_1 and C_2 such that $e^{\frac{-c_1}{2}} c_1^{\frac{n}{2}} = e^{\frac{-c_2}{2}} c_2^{\frac{n}{2}} = \frac{\lambda_\alpha}{k}$

And $P[\chi^2(n-1) \le c_1] + P[\chi^2(n-1) \ge c_2] = \alpha$ where $c_1 = \chi_{1-\alpha/2}{}^2(n-1)$ and $C_2 = \chi_{\alpha/2}{}^2(n-1)$

In practice to test the null hypothesis

H_0: $\sigma=\sigma_0$ against H_1: $\sigma \neq \sigma_0$, $\frac{\sum_i (xi-\bar{x})^2}{\sigma_0{}^2}$ is computed and compared with the tabulated value.

That is we reject the null hypothesis if $\frac{\sum_i (xi-\bar{x})^2}{\sigma_0{}^2} > \chi_{\alpha/2}{}^2(n-1)$ or

$$\frac{\sum_i (xi-\bar{x})^2}{\sigma_0{}^2} < \chi_{1-\alpha/2}{}^2(n-1)$$

14. Derive LRTP to test the null hypothesis H0: $\sigma=\sigma_0$ against H1: $\sigma>\sigma_0$, with an unknown mean μ

Answer:

Let x_1, $x_2 \ldots x_n$ be a random sample of size n from a Normal population with parameters μ and σ^2

$$\lambda = \frac{\underset{\mu,\sigma=\sigma_0}{Sup} \; L(x1, x2, \ldots., xn)}{\underset{\mu,\sigma\ge 0}{Sup} \; L(x1, x2, \ldots., xn)}$$

$$P[\lambda \le \lambda_\alpha / H_0] = \alpha - - - - - - (*)$$

Where λα is such that

$$\lambda = \frac{\underset{\mu,\sigma \;=\sigma_0}{Sup} \; (\frac{1}{\sigma^2})^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}{\underset{\mu,\sigma \;\ge 0}{Sup} \; (\frac{1}{\sigma^2})^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu \;)^2}}$$

Under the null hypothesis the m.l.e of μ = $\bar{x}$ otherwise the m.l.e's of μ and σ^2 are $\bar{x}$ and $\frac{\sum (xi-\bar{x})^2}{n}$

In the expression of λ substituting the parameters by m.l.e's so that numerator and denominator attains its supremum we have

$$\lambda = \frac{\left(\frac{1}{\sigma_0^{2}}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n} e^{\frac{-1}{2\sigma_0^{2}} \sum_i (xi-\bar{x})^2}}{\left(\frac{1}{\frac{\sum (xi-\bar{x})^2}{n}}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n} e^{\frac{-1}{2\frac{\sum (xi-\bar{x})^2}{n}} \sum_i (xi-\bar{x})^2}}$$

$$\Rightarrow \left(\frac{\sum (xi-\bar{x})^2}{\sigma_0^2}\right)^{\frac{n}{2}} n^{\frac{-n}{2}} e^{\frac{n}{2}} e^{\left(\frac{-1}{2}\right)\frac{\sum (xi-\bar{x})^2}{\sigma_0^2}}$$

$$\Rightarrow e^{\frac{-u}{2}} u^{\frac{n}{2}} k \text{ where } u = \frac{\sum (xi-\bar{x})^2}{\sigma_0^2}, k = n^{\frac{-n}{2}} e^{\frac{n}{2}}$$

$$\lambda \le \lambda_\alpha \Rightarrow \frac{\sum_i (xi-\bar{x})^2}{\sigma_0^{2}} > C_1 = \chi_\alpha^{2}(n-1)$$

Now size of the test =α $\Rightarrow P[\text{Re}\, ject H_0 / H_0 true] = \alpha$

$$\Rightarrow P[\lambda \le \lambda_\alpha / \sigma = \sigma_0] = \alpha$$

$$\Rightarrow P\left[\frac{\sum_i (xi-\bar{x})^2}{\sigma_0^{2}} > \chi_\alpha^{2}(n-1)\right] = \alpha$$

In practice to test the null hypothesis

H0: $\sigma=\sigma_0$ against H1: $\sigma>\sigma_0$, the mean is unknown $\frac{\sum_i (xi-\bar{x})^2}{\sigma_0^{2}}$ is computed and compared with the tabulated value $\chi_\alpha^{2}(n-1)$.

That is we reject the null hypothesis if $\frac{\sum_i (xi-\bar{x})^2}{\sigma_0^{2}}$ exceeds $\chi_\alpha^{2}(n-1)$

15. Derive LRTP to test the null hypothesis H0: $\sigma=\sigma_0$ against H1: $\sigma<\sigma_0$, with an unknown mean μ

Answer:

Let $x_1, x_2 \ldots x_n$ be a random sample of size n from a Normal population with parameters μ and σ^2

$$\lambda = \frac{\underset{\mu,\sigma=\sigma_0}{Sup} \; L(x1, x2, ..., xn)}{\underset{\mu,\sigma\geq 0}{Sup} \; L(x1, x2, ..., xn)}$$

Where λα is such that

$$P[\lambda \leq \lambda_\alpha / H_0] = \alpha ------(*)$$

$$\lambda = \frac{\underset{\mu,\sigma=\sigma_0}{Sup} \; (\frac{1}{\sigma^2})^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}{\underset{\mu,\sigma\geq 0}{Sup} \; (\frac{1}{\sigma^2})^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma^2}\sum_i (xi-\mu)^2}}$$

Under the null hypothesis the m.l.e of μ = $\overline{x}$ otherwise the m.l.e's of μ and σ^2 are $\overline{x}$ and $\frac{\sum (xi - \overline{x})^2}{n}$.

In the expression of λ substituting the parameters by m.l.e's so that numerator and denominator attains its supremum we have

$$\lambda = \frac{(\frac{1}{\sigma_0^2})^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{-1}{2\sigma_0^2}\sum_i (xi-\overline{x})^2}}{\left(\frac{1}{\frac{\sum (xi-\overline{x})^2}{n}}\right)^{\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n e^{\frac{\frac{-1}{\sum (xi-\overline{x})^2}\sum_i (xi-\overline{x})^2}{2\frac{}{n}}}}$$

$$\Rightarrow \left(\frac{\sum (xi-\overline{x})^2}{\sigma_0^2}\right)^{\frac{n}{2}} n^{\frac{-n}{2}} e^{\frac{n}{2}} e^{(\frac{-1}{2})\frac{\sum (xi-\overline{x})^2}{\sigma_0^2}}$$

$$\Rightarrow e^{\frac{-u}{2}} u^{\frac{n}{2}} k \text{ where } u = \frac{\sum (xi-\overline{x})^2}{\sigma_0^2}, k = n^{\frac{-n}{2}} e^{\frac{n}{2}}$$

Now

$$\lambda \le \lambda_\alpha \Rightarrow \frac{\sum_i (xi - \bar{x})^2}{\sigma_0{}^2} < C_2 = \chi_{1-\alpha}{}^2 (n-1)$$

Now size of the test =α $\Rightarrow P[\text{Re}\, ject H_0 / H_0 true] = \alpha$

$$\Rightarrow P[\lambda \le \lambda_\alpha / \sigma = \sigma_0] = \alpha$$

$$\Rightarrow P[\frac{\sum_i (xi - \bar{x})^2}{\sigma_0{}^2} < \chi_{1-\alpha}{}^2 (n-1)] = \alpha$$

In practice to test the null hypothesis

H0: σ=σ_0 against H1: σ<σ_0, the mean is unknown $\frac{\sum_i (xi - \bar{x})^2}{\sigma_0{}^2}$ is computed and compared with the tabulated value $\chi_{1-\alpha}{}^2 (n-1)$.

That is we reject the null hypothesis if $\frac{\sum_i (xi - \bar{x})^2}{\sigma_0{}^2}$ is less than $\chi_{1-\alpha}{}^2 (n-1)$