1. Introduction

Welcome to the series of E-learning modules on Applications of MP test and Best Critical Region. In this module we are going cover the procedure to obtain the Best critical region, Most Powerful tests and application of the procedure to Normal population.

By the end of this session, you will be able to:

- Explain the Procedure to obtain a Best critical region
- Describe Most Powerful tests
- Explain the application of the procedure to Normal population

Let us first recollect the basic concepts of Best Critical Region and Most Powerful test and then look into the procedure to obtain a Best Critical Region. Let us focus on the testing the simple null hypothesis against a simple alternative hypothesis.

Let a simple null hypothesis that theta equal to theta naught be tested against the simple alternative hypothesis, theta equal to theta one.
Let C be a critical region for testing H naught of size alpha against H one.
C is called the best critical region of size alpha for testing H NAUGHT against H ONE if C has at least the same power as any other critical region of size alpha. That is, if C is a critical region of size alpha and C dash is any other critical region of size alpha then Power of C greater than or equal to Power of C dash.

Most Powerful Test:
Among the critical regions of the same size alpha that which renders the *minimum* Type two error is called the *most powerful critical region*.
The test based on the most powerful critical region is called the most powerful test.
Therefore among all tests possessing the same size of Type one error, choose one for which the size of the Type two error is small as possible. This test is called the Most Powerful test.

Neyman- Pearson Lemma for finding out the best test
Let the probability density function of the population be f of (x, theta) where theta represents the parameters.
Draw a sample (x one, x two, etc x n) from this population.
Let L one represent the likelihood function when H one is true and L naught represent the likelihood function when H naught is true.
H naught and H one are simple hypotheses.

Let alpha be the level of significance. Let k be any constant such that the size of the critical region defined as below is alpha.
C equal to { (x one, x two, etc till x n): L one of (x i , theta) by L naught of (x i , theta) is greater than k}
Then *C* is the best, that is, most powerful, critical region for testing the simple null hypothesis H *naught that* theta is equal to theta naught against the simple alternative hypothesis H *one that* theta is equal to theta one.

Now let us see how to obtain an Most Powerful test or a Best Critical region using NP lemma while testing a hypothesis related to some standard populations like Normal.

2. Example 1

Procedure to obtain a Best Critical Region

- Obtain likelihood functions under null hypothesis and alternative hypothesis and call them as L naught and L one
- Obtain the ratio L one by L naught
- Obtain the critical region as a region satisfying the condition L one by L naught greater than k, where k is determined such that Probability of x belongs to the above mentioned Critical region under null hypothesis is equal to the size of the test alpha

Example one

Find the Best Critical Region of size alpha for testing the null hypothesis that mu is equal to mu naught against the alternative hypothesis that mu is equal to mu one when a sample of size 'n' is drawn from the Normal population with a known variance sigma square.

Solution:
Let x one, x two, etc till xn be a random sample of size n from a Normal population with parameters mu and sigma square.
By NP Lemma a size alpha Best Critical Region is given is given by: C is equal to { (x one, x two, etc till x n): L one of (x i , mu) by L naught of (x i , mu) is greater than k.
Where k is such that Probability of x belongs to C given that H naught is true is equal to alpha. Call this as equation one.

L one of (x i, mu) by L naught of (x i, mu) is equal to one by sigma into root two pi to the power n into e to the power minus one by sigma square into summation (x i minus mu one) whole square divided by one by sigma into root two pi to the power n into e to the power minus one by sigma square into summation (x i minus mu naught) whole square.
Which is equal to e to the power minus one by two sigma square into [summation (x i minus mu one) the whole square minus summation (xi minus mu naught) the whole square]
Which implies e to the power minus one by two sigma square into [summation x i square plus n mu one square minus two mu one summation x i minus (summation x i square plus n mu naught square minus two mu naught into summation x i)]
This implies e to the power minus one by two sigma square into [n into (mu one square minus mu naught square) minus two into summation x i into (mu one minus mu naught)]

Now, L one of (x i , mu) by L naught of (x i , mu) greater than k implies 'e' to the power minus one by two sigma square into [n into (mu one square minus mu naught square) minus two into summation x i into (mu one minus mu naught)] is greater than k.
This implies minus one by two sigma square into [n into (mu one square minus mu naught square) minus two into summation x i into (mu one minus naught)] greater than Ln k.
Which implies minus one by two sigma square into [n into (mu one square minus mu naught square) minus two into 'n' into x bar into (mu one minus naught)] is greater than Ln k.
This implies, x bar into (mu one minus naught) greater than[Ln k plus n by two sigma square into (mu one square minus mu naught square)]the whole multiplied by sigma square by 'n' which is equal to lambda one (say)

3. Case 1 and Case 2 of Example 1

Case one:
Let mu one be greater than mu naught
L one of (x i , mu) by L naught of (x i , mu) greater than k implies x bar greater than lambda one by (mu one minus mu naught)
X bar minus mu naught by sigma by root n is greater than lambda one by (mu one minus mu naught) minus mu naught divided by sigma by root n is equal to lambda two say. Call this as equation two.
When xi follows normal with parameters mu and sigma square, a sample mean x bar follows normal with mean mu and variance sigma square by n.
Then x bar minus mu naught by sigma by root n follows Normal distribution with mean zero and variance one under Null hypothesis.

By substituting equation (two) in equation (one) we get
Probability of x belongs to C given H naught true is equal to alpha which implies Probability of L one of (x i , mu) by L naught of (x i , mu) is greater than k given H naught is true is equal to alpha.
Which implies Probability of x bar minus mu naught by sigma by root n is greater than lambda two is equal to alpha implies probability of Z greater than lambda two is equal to alpha.

Now from the table of Normal Probabilities lambda square can be read. Using the relation among lambda one, lambda two and k, k can be determined. Hence the BCR can be found.
Hence the BCR when mu one is greater than mu naught is given by, C is equal to {(x one, x two, etc till x n): Z greater than lambda two which is equal to Z alpha by two}.

Case 2:
Let mu one be less than mu naught
L one of (x i , mu) by L naught of (x i , mu) less than k implies x bar less than lambda one by (mu one minus mu naught).
X bar minus mu naught by sigma root n is less than lambda one by mu one minus mu naught divided by sigma by root n is equal to lambda two say. Call this as equation three.

By substituting equation (3) in equation (one) we get
Probability of L one of (x i , mu) by L naught of (x i , mu) less than k given H naught is true is equal to alpha.
Which implies Probability of x bar minus mu naught by sigma by root n is less than lambda two is equal to alpha implies probability of Z less than lambda two is equal to alpha.
Hence when mu one is less than mu naught the Best Critical Region is given by C is equal to { (x one, x two, etc till x n): Z less than lambda two which is equal to minus Z alpha by two}.

4. Example 2

Example two
Find the BCR of size alpha for testing the null hypothesis H naught that: sigma is equal to sigma naught against the alternative hypothesis H one that sigma is equal to sigma one when a sample of size 'n' is drawn from the Normal population with a known mean mu.

Solution:
Let x one, x two, etc x n be a random sample of size n from a Normal population with parameters mu and sigma square.
By NP Lemma a size alpha BCR is given is given by,
C is equal to { (x one, x two, etc x n): L one of (x i , sigma square) by L naught of (x i , sigma square) greater than k.
Where k is such that Probability of x belongs to C given that H naught is true is equal to alpha.

L one of (x i , sigma square) by L naught of (x i , sigma square) is equal to one by sigma one square to the power n by two into one by root two pi to the power n into 'e' to the power minus one by two sigma one square into summation (x i minus mu)the whole square, the whole divided by one by sigma naught square to the power 'n' by two into one by root two pi to the power n into e to the power minus one by two sigma naught square into summation (x i minus mu) whole square.
Which implies sigma naught square by sigma one square to the power n by two into e to the power minus one by two into (one by sigma one square minus sigma naught square) into summation (xi minus mu) the whole square.

Now, L one of (x i , sigma square) by L naught of (x i , sigma square) greater than k implies sigma naught square by sigma one square to the power n by two into e to the power minus one by two into (one by sigma one square minus one by sigma naught square) into summation (x i minus mu) whole square greater than k.
Which implies n by two into Ln of sigma naught square by sigma one square minus one by two into (one by sigma one square minus one by sigma naught square) into summation (x i minus mu) the whole square greater than Ln k.
Which implies by 'n' two into Ln of sigma naught square by sigma one square minus one by two into (sigma naught square minus sigma one square) by sigma one square into sigma naught square into summation (x i minus mu) whole square is greater than Ln k.

Which implies n by two into Ln of sigma naught square by sigma one square plus one by two into (sigma one square minus sigma naught square) by sigma one square into sigma naught square into summation (x i minus mu) the whole square is greater than Ln k.
Which implies one by two into (sigma one square minus sigma naught square) by sigma one square into sigma naught square into summation (x i minus mu) whole square is greater than Lnk minus n by two into Ln of sigma naught square by sigma one square.
Which implies (sigma one square minus sigma naught square) into summation (x i minus mu) whole square is greater than[ln k minus n by two into Ln of sigma naught square by sigma one square] into two sigma one square into sigma naught square which is equal to lambda one, (say)

5. Case 1 and Case 2 of Example 2

Case 1:
Let sigma one be greater than sigma naught
L one of (x i , sigma square) by L naught of (x i , sigma square) greater than k implies summation (x i minus mu) whole square is greater than lambda one by (sigma one square minus sigma naught square).
This implies, summation (x i minus mu) whole square by sigma naught square is greater than lambda one by (sigma one square minus sigma naught square) by sigma naught square equal to lambda two, say.
Now size of the test is equal to alpha, which implies, Probability of L one Of xi coma sigma square by L naught of xi coma sigma square is greater than k given sigma is equal to sigma naught is equal to alpha. Call this equation as star.

When xi follows normal, with parameters mu and sigma square, and when sigma is equal to sigma naught, summation (xi minus mu), whole square by sigma naught square follows chi square variable with n degrees of freedom.
Then from the table of probabilities of Chi square variable with n degrees of freedom, we have lambda two is equal to chi square alpha with n degrees of freedom.

By substituting equation (one) in equation (star) we get,
Probability of x belongs to C given H naught true is equal to alpha implies, probability of L one of (x i coma sigma square) by L naught of (x i coma sigma square) greater than k given sigma is equal to sigma naught is equal to alpha.
This implies probability of summation (x i minus mu) whole square by sigma naught square greater than lambda two, which implies probability of summation (x i minus mu) whole square by sigma naught square greater than chi square alpha with n degrees of freedom is equal to alpha.

Now from the table of Chi square probabilities lambda two can be read.
Using the relation among lambda one, lambda two and k, k can be determined. Hence the Best Critical Region can be found.
The best critical region when sigma one greater than sigma naught is given by,
C is equal to {(x one, x two, etc x n) colon, summation (x i minus mu) the whole square by sigma naught square is greater than Chi square alpha with ‘n’ degrees of freedom}.

Case 2:
Let sigma one be less than sigma naught
L one of (x i , sigma square) by L naught of (x i , sigma square) less than k implies summation (x i minus mu) the whole square is less than lambda one by (sigma one square minus sigma naught square).
This implies summation (x i minus mu) whole square by sigma naught square is less than lambda one by (sigma one square minus sigma naught square) by sigma naught square is equal to lambda two, say.

Now size of the test is equal to alpha implies, probability of L one of (xi coma sigma square) by L naught of (xi coma sigma square) is less than K given (sigma is equal to sigma naught), is equal to alpha. Call this as equation star.

When xi follows normal, with parameters mu and sigma square, and when sigma is equal to sigma naught, summation (xi minus mu), whole square by sigma naught square follows chi square variable with n degrees of freedom.
From the table of probabilities of Chi square variable with n degrees of freedom we have lambda two is equal to Chi square (one minus alpha) with n degrees of freedom.

By substituting equation (one) in equation (star) we get, probability of x belongs to C given H naught is true, is equal to alpha implies, probability of L one of (x i , sigma square) by L naught of (x i , sigma square) less than k given sigma is equal to sigma naught is equal to alpha.

Which implies probability of summation (x i minus mu) whole square by sigma naught square is less than lambda two, implies, probability of summation (x i minus mu) the whole square by sigma naught square is less than chi square is equal to chi square (one minus alpha) with n degrees of freedom is equal to alpha.
The Best Critical Region when sigma one is less than sigma naught is given by,
C equal to { (x one, x two, etc till x n) colon, summation (x i minus mu) the whole square by sigma naught square is less than Chi square (one minus alpha)with n degrees of freedom}.

Here's a summary of our learning in this session where we have understood the following concepts:

- Best Critical region and Most Powerful test
- Procedure to obtain MP test
- MP test for testing the hypothesis related to normal population