## Frequently Asked Questions

1. Briefly explain a likelihood function?

**Answer:**

If $X_1$, $X_2$, ..., $X_n$ is a random sample of size $n$ from a distribution with probability density (or mass) function $f(x;\theta)$, then the joint probability density (or mass) function of $X_1$, $X_2$, ..., $X_n$ is denoted by the likelihood function $L(\theta)$. That is, the joint p.d.f. or p.m.f. is:

$$L(\theta)=L(\theta;x1,x2,...,xn)=f(x1;\theta)\times f(x2;\theta)\times...\times f(xn;\theta)$$

Note that for the sake of ease, we drop the reference to the sample $x_1$, $x_2$... $x_n$ in using $L$ ($\theta$) as the notation for the likelihood function. We'll want to keep in mind though that the likelihood $L$ ($\theta$) still depends on the sample data.

2. How do you distinguish between Simple and Composite Hypothesis?

**Answer:**

If a random sample is taken from a distribution with parameter $\theta$, a hypothesis is said to be a simple hypothesis if the hypothesis *uniquely specifies* the distribution of the population from which the sample is taken. Any hypothesis that is not a simple hypothesis is called a composite hypothesis**.**

3. What is a Best Critical Region?

**Answer:**

Let a simple null hypothesis $H_0$: $\theta=\theta_0$ be tested against the simple alternative hypothesis $H_1$: $\theta=\theta_1$ . Let C be a critical region for testing $H_0$ of size α against $H_1$. C is called the best critical region of size α for testing $H_0$ against $H_1$ if C has at least the same power as any other critical region of size α that is if C is a critical region of size α and C' is any other critical region of size α then Power of C greater than or equal to Power of C' .

In testing of hypothesis we keep α the level of significance at a fixed level ( say 0.05 or 0.01) and try to minimize the Type II error. The sample space may be partitioned into several ways so that each critical region, w has the same size α. Of all these critical regions choose that which has least type II error. This is called the best critical region of size α.

4. How do you define a Most Powerful (MP) test?

**Answer:**

Among the critical regions of the same size α that which renders the minimum Type II error is called the most powerful critical region. The test based on the most powerful critical region is called the most powerful test. Therefore among all tests possessing the same size of Type I error, choose one for which the size of the Type II error is small as possible. This test is called the Most Powerful test.

Hence if two tests have the same level of significance, then the test with a smaller-size type II error is the most powerful test of the two at that significance level.

5. State Neyman- Pearson Lemma for finding out the best test.

**Answer:**

Let the probability density function of the population be f(x,θ) where θ represents the parameters. Draw a sample (x1,x2,…,xn) from this population. Let L be the likelihood function then L=f(x1,x2,…,xn). Since the sample is drawn independently f(x1,x2,…,xn)= f(x1) f(x2)….f(xn)

Let $L_1$ stands for the likelihood function when $H_1$ is true and $L_0$ stands for the likelihood function when $H_0$ is true ( $H_0$ and $H_1$ are simple hypothesis)

Lo= $f(x1/H_0)$ . $f(x2/H_0)$ …….. $f(xn/H_0)$

$L_1$= $f(x1/H_1)$ . $f(x2/H_1)$ …….. $f(xn/H_1)$ . Let α be the level of significance

Let w be the critical region. So w is a subset of the sample space. Then the theorem states that we can determine w in such a way that $L_1/ L_0 >k$ where k is the value determined on the basis of the level of significance or size of the test. Then w is called a Best Critical region or the most powerful critical region of the significant level α for testing $H_0$ against $H_1$. The test based on the most powerful critical region is called the most powerful test. *The Neyman-Pearson lemma* gives sufficient conditions for a best critical region of size α.

**6.** Distinguish between MP tests and UMP tests?

**Answer:**

A Neyman Pearson lemma gives BCR for testing a simple null hypothesis $H_0 : \theta = \theta_0$ against simple alternative hypothesis $H_1{:}\theta{=}\theta_1$. The test based on the most powerful critical region or Best critical region given by NP lemma is called the most powerful test.

Suppose T is a test of size α for testing $H_0$ : θ belongs to the parameter space under $H_0$ against an alternative hypothesis $H_1$: θ belongs to the parameter space under $H_1$. T is said to a uniformly most powerful test (UMPT) of size α for testing $H_0$ against $H_1$ if for any other test T' of size α for testing $H_0$ against $H_1$ , $P_T(\theta) > P_{T'}(\theta)$ for all θ belongs to the parameter space under $H_1$ where $P_T(\theta)$ denotes the power of the test

Hence When we are testing a simple null hypothesis versus a simple alternative hypothesis we use most powerful test and in case of a simple null hypothesis versus a composite alternative hypothesis we apply Uniformly most powerful test.

**7.** When do UMP tests exist?

**Answer:**

- Hypothesis pair concerns a single real-valued parameter
- Hypothesis pair is "one-sided"

However existence of UMP test does not only depend on hypothesis also depends on specific model.

**8.** Define an Uniformly most powerful test (UMPT)?

**Answer:**

Suppose T is a test of size α for testing H0: θ belongs to the parameter space under $H_0$ against an alternative hypothesis $H_1$: θ belongs to the parameter space under $H_1$. T is said to a uniformly most powerful test (UMPT) of size α for testing $H_0$ against $H_1$ if for any other test T' of size α for testing $H_0$ against $H_1$ , $P_T(\theta) > P_{T'}(\theta)$for all θ belongs to the parameter space under $H_1$ where $P_T(\theta)$ denotes the power of the test

The Neyman-Pearson Lemma can be used in certain cases to derive optimal tests of a simple null versus a composite alternative.

In statistical hypothesis testing, a uniformly most powerful (UMP) test is a hypothesis test which has the greatest power $1 - \beta$ among all possible tests of a given size $\alpha$. For example, according to the Neyman–Pearson lemma, the likelihood-ratio test is UMP for testing simple (point) hypotheses.

**9.** What is uniformly most powerful critical region?

**Answer:**

The region C is a uniformly most powerful critical region of size α for testing the simple hypothesis H against a composite alternative hypothesis $H_1$ if C is a best critical region of size α for testing $H_0$ against each simple hypothesis in $H_1$. The resulting test is said to be uniformly most powerful.

**10.** What do you mean by an unbiased test procedure and critical region?

**Answer:**

The test procedure for which the power is more than its size is called an unbiased test. That is if (1 – ß) > α then it is an unbiased test procedure. Similarly a critical region whose power is more than its size is called unbiased critical region.

**11.** Define a consistent test or consistent critical region?

**Answer:**

In a test procedure if the power tends to 1 as n tends to infinity then the test procedure is called consistent test procedure that is (1 – ß) → 1 as n → ∞. Similarly a critical region whose power tends to 1 as n tends to infinity then it is said to be a consistent critical region

**12.** In testing a simple null hypothesis against a simple alternative hypothesis the power of the Best Critical Region cannot be less than its size or BCR obtained by NP lemma is unbiased

**Answer:**

Let $x_1, x_2 \ldots x_n$ be a sample of size n taken from the population f(x,θ). Using a NP lemma the BCR is given by

$$C = \left\{ (x1, x2, ..., xn) : \frac{L_1(xi,\theta)}{L_0(xi,\theta)} \geq k \right\}$$

Therefore Acceptance region is given by,

$$A = \left\{ (x1, x2, ..., xn) : \frac{L_1(xi,\theta)}{L_0(xi,\theta)} < k \right\}$$

$\Rightarrow L_1(xi,\theta) < kL_0(xi,\theta)$ in the acceptance region

And $L_1(xi,\theta) \geq kL_0(xi,\theta)$ in the rejection region

Consider:

$L_1(xi,\theta) \geq kL_0(xi,\theta) \Rightarrow L_1 \geq kL_0$ where $L_1 = L_1(xi,\theta)$ and $L_0 = L_0(xi,\theta)$

$$\Rightarrow \int_C L_1 dx \geq k \int_C L_0 dx$$

$$\Rightarrow P[\mathrm{Re}\, jectH_0 / H_1] \geq kP[\mathrm{Re}\, jectH_0 / H_0]$$

$$\Rightarrow (1-\beta) \geq k\alpha ------(1)$$

Consider $\dfrac{L_1(xi,\theta)}{L_0(xi,\theta)} < k$

$$\Rightarrow L_1(xi,\theta) < kL_0(xi,\theta) \Rightarrow L_1 < kL_0$$
$$\Rightarrow \int_A L_1 dx < k\int_A L_0 dx$$
$$\Rightarrow P[AcceptH_0 / H_1] < kP[AcceptH_0 / H_0]$$
$$\Rightarrow \beta < k(1-\alpha) ------(2)$$
Equation (1) multiplied by (-1) gives
$$-(1-\beta) < -k\alpha ------(3)$$
Equation (3) / (2) gives
$$-\frac{(1-\beta)}{\beta} < \frac{-k\alpha}{k(1-\alpha)}$$

$$\Rightarrow -[(1-\beta)k(1-\alpha)] < -k\alpha\beta$$
$$\Rightarrow [\alpha-(1-\beta)] < 0 \Rightarrow (1-\beta) > \alpha$$
Hence BCR obtained by NP lemma is unbiased

**13.** Obtain a BCR of size α for testing H0: λ=λ0 against H1: λ=λ1(<λ0 )when a sample of size n is drawn from the Poisson population with unknown parameter λ?

**Answer:**

Let $x_1, x_2 \ldots x_n$ be a random sample of size n from Poisson population with parameter λ

By NP Lemma a size α BCR is given is given by

$$C = \left\{(x1, x2, ..., xn) : \frac{L_1(xi,\lambda)}{L_0(xi,\lambda)} > k\right\}$$
$$P[x \in C / H_0] = \alpha$$

Where k is such that

$$\frac{L_1(xi,\lambda)}{L_0(xi,\lambda)} = \frac{\dfrac{e^{-n\lambda_1}\lambda_1^{\sum_i x_i}}{\prod_{i=}^{n} x_i!}}{\dfrac{e^{-n\lambda_0}\lambda_0^{\sum_i x_i}}{\prod_{i=}^{n} x_i!}} = e^{-n(\lambda_1-\lambda_0)}\left(\frac{\lambda_1}{\lambda_0}\right)^{\sum_i x_i}$$

Now $\dfrac{L_1(xi,\lambda)}{L_0(xi,\lambda)} > k \Rightarrow e^{-n(\lambda_1-\lambda_0)}\left(\dfrac{\lambda_1}{\lambda_0}\right)^{\sum_i x_i} > k$

$$\Rightarrow -n(\lambda_{1}-\lambda_{0}) + \sum_{i} xi \ln\left(\frac{\lambda_1}{\lambda_0}\right) > \ln k$$

$$\Rightarrow \sum_{i} xi \ln\left(\frac{\lambda_1}{\lambda_0}\right) > \ln k + n(\lambda_{1}-\lambda_{0}) = k_1 (say)$$

When λ1 < $\lambda_0$

$$\frac{L_1(xi,\lambda)}{L_0(xi,\lambda)} < k \Rightarrow \sum_{i} xi < \frac{k_1}{\ln\left(\frac{\lambda_1}{\lambda_0}\right)} = k_2$$

The BCR when $\lambda_1 < \lambda_0$ is given by $C = \left\{(x1, x2,..., xn) : \sum_{i} x_i < k_2\right\}$

**14.** Find the BCR of size α for testing H0: λ=λ0 against H1: λ=λ1(>λ0 )when a sample of size n is drawn from the Poisson population with unknown parameter λ?

**Answer:**

Let $x_1$, $x_2$,…,$x_n$ be a random sample of size n from Poisson population with parameter λ

By NP Lemma a size α BCR is given is given by

$$C = \left\{(x1, x2,..., xn) : \frac{L_1(xi,\lambda)}{L_0(xi,\lambda)} > k\right\}$$

$$P[x \in C / H_0] = \alpha$$

Where k is such that

$$\frac{L_1(xi,\lambda)}{L_0(xi,\lambda)} = \frac{\dfrac{e^{-n\lambda_1}\lambda_1^{\sum_i x_i}}{\prod_{i=}^{n} x_i!}}{\dfrac{e^{-n\lambda_0}\lambda_0^{\sum_i x_i}}{\prod_{i=}^{n} x_i!}} = e^{-n(\lambda_1-\lambda_0)}\left(\frac{\lambda_1}{\lambda_0}\right)^{\sum_i x_i}$$

Now $\dfrac{L_1(xi,\lambda)}{L_0(xi,\lambda)} > k \Rightarrow e^{-n(\lambda_1-\lambda_0)}\left(\dfrac{\lambda_1}{\lambda_0}\right)^{\sum_i x_i} > k$

$$\Rightarrow -n(\lambda_{1}-\lambda_{0}) + \sum_{i} xi \ln\left(\frac{\lambda_1}{\lambda_0}\right) > \ln k$$

$$\Rightarrow \sum_i xi \ln\left(\frac{\lambda_1}{\lambda_0}\right) > \ln k + n(\lambda_{1-}\lambda_0) = k_1 (say)$$

When $\lambda_1 > \lambda_0$

$$\frac{L_1(xi,\lambda)}{L_0(xi,\lambda)} > k \Rightarrow \sum_i xi > \frac{k_1}{\ln\left(\frac{\lambda_1}{\lambda_0}\right)} = k_2$$

The BCR when $\lambda_1 > \lambda_0$ is given by $C = \left\{(x1, x2, ..., xn) : \sum_i x_i > k_2\right\}$

**15.** If x1,x2,…,xn is a random sample of size n from a distribution having probability density function of the form $f(x,\theta) = \theta x^{\theta-1}$

Show that a BCR for testing H0:θ=1 against H1:θ=2 is

$$C = \left\{(x1, x2, ..., xn) : \prod_j x_j \geq c\right\}$$

**Answer:**

Let $x_1$, $x_2$… $x_n$ be a random sample of size n from population with probability density function of the form $f(x,\theta) = \theta x^{\theta-1}$

Required to test H0:θ=1 against H1:θ=2

By NP Lemma a size α BCR is given is given by

$$C = \left\{(x1, x2, ..., xn) : \frac{L_1(xi,\theta)}{L_0(xi,\theta)} \geq k\right\}$$

$$P[x \in C / H_0] = \alpha$$

Where k is such that

$$L(xj,\theta) = \prod_j f(xj,\theta) = \theta^n \prod_j x_j^{\theta-1}$$

$$\frac{L_1(xi,\theta)}{L_0(xi,\theta)} = \frac{\theta_1^n \prod_j x_j^{\theta_1 - 1}}{\theta_0^n \prod_j x_j^{\theta_0 - 1}} = \frac{2^n \prod_j x_j^{2-1}}{1^n \prod_j x_j^{1-1}} = 2^n \prod_j x_j$$

$$\frac{L_1(xi,\theta)}{L_0(xi,\theta)} \geq k \Rightarrow 2^n \prod_j x_j \geq k \Rightarrow \prod_j x_j \geq \frac{k}{2^n} = c$$

A size α BCR is given by $C = \left\{(x1, x2, ..., xn) : \prod_j x_j \geq c\right\}$