Frequently Asked Questions

1. What do you mean by method of moments?

Answer:

The method of moments is the oldest method of deriving point estimators. It almost always produces some asymptotically unbiased estimators, although they may not be the best estimators.

The method was moments were discovered by Karl Pearson. As the name itself suggests in this technique moments are utilized for the estimation of the unknown parameters. This method is based on the principle that the unknown population parameters can be estimated by making use of sample moments. By equating the sample moments to the population raw moments the unknown parameters of the population may be estimated.

2. Write a note on the principle behind the method of moments.

Answer:

The method of moments is based on the moments of population as well as that of sample.

Let f(x, θ1, θ2... θ_k) is the density function of the population under consideration. To estimate the unknown parameters θ_1, θ2, θ_k if μ_r' denotes the r the moment (about zero) then by definition

$\mu_r' = \int x^r f(x, \theta 1, \theta 2 \ldots \theta_k)\, dx$, r=12…

That is

$\mu_1' = \int x\, f(x, \theta 1, \theta 2 \ldots \theta_{k})\, dx$

$\mu_2' = \int x^2 f(x, \theta 1, \theta 2 \ldots \theta_k)\, dx$

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$\mu_{k'} = \int x^k f(x, \theta 1, \theta_2, \ldots, \theta_k)\, dx$

$\mu_1', \mu_2', \mu_3', \ldots, \mu_k'$ are in general functions of the parameters $\theta_1, \theta_2, \ldots, \theta_k$. Thus the above is a set of k equations involving k unknown parameters $\theta_1, \theta_2, \ldots, \theta_k$.

Now solving the equations $\theta_1, \theta_2, \ldots, \theta_k$ can be written as the functions of $\mu_1', \mu_2', \mu_3', \ldots, \mu_k'$. But in general $\mu_1', \mu_2', \mu_3', \ldots, \mu_k'$ are unknown and hence their estimators by sample moments $m_1', m_2', m_3', \ldots, m_k'$ respectively where $m_{r'} = \sum x_i^r /n$, x1,x2,…xn being sample observations are determined.

In case of frequency distribution of the sample observations $r^{th\ sample}$ moment m_r1 is given by $m_{r'} = \sum_{fixi}{}^r / N$. Thus the method of moments consists in equating r^{th} raw moments about the origin in the population to the r the raw moments about the origin in the sample by giving values r=1,2,…. and obtaining various equations containing parameters and solving these equations to obtain the estimate of the parameters.

3. What are the properties of moment estimators?

Answer:

The basic properties of moment estimators are

- Moment estimators are consistent provided the population moments exists
- Moment estimators need not be unbiased
- Under certain conditions moment estimators have an asymptotic Normal distribution
- Moment estimators are less efficient than maximum likelihood estimators

4. Why the sample moments are consistent estimators of the corresponding population moments?
Answer:

Let x1,x2,…,xn be a random sample of size n from a population with probability density function(x, θ). Then xi (i=1,2,..,n) are i.i.d . Hence if $E(x_i^r)$ exists then by W.L.L.N we get

$$\frac{1}{n}\sum_{i=1}^{n} x_i^r \xrightarrow{P} E(x_i^r) \Rightarrow m_r' \xrightarrow{P} \mu_r'$$

Hence sample moments are consistent estimators of the corresponding population moments

5. What are the advantages of method of moments?
Answer:
In some respects, when estimating parameters of a known family of probability distributions, this method of moments was superseded by Fisher's method of maximum likelihood, because maximum likelihood estimators have higher probability of being close to the quantities to be estimated.

However, in some cases, as in the population of the gamma distribution, the likelihood equations may be intractable without computers, whereas the method-of-moments estimators can be quickly and easily calculated by hand.

Estimates by the method of moments may be used as the first approximation to the solutions of the likelihood equations, and successive improved approximations may then be found by the Newton–Raphson method. In this way the method of moments and the method of maximum likelihood are symbiotic.

When estimating other structural parameters (e.g., parameters of a utility function, instead of parameters of a known probability distribution), appropriate probability distributions may not be known, and moment-based estimates may be preferred to Maximum Likelihood Estimation.

Method of moments has been used to solve vast number of electromagnetic problems during the last five decades. The method of methods has been used in electromagnetic theory The Method of Moments technique, as applied to problems in electromagnetic theory, was introduced by Roger F.Harrington in his 1967seminar paper, "Matrix Methods for Field Problems". The implementation of the Method of Moments, by Poggio and Burke at Lawrence Livermore National Labs during the 1970s, established this solution technique as a mainstay in the design of wire and wire array antennas. They can be used to solve closed or open models and also to solve in frequency (the most common) and in time domain.

6. What are the limitations of the method of moments?
Answer:
In some cases, infrequent with large samples but not so infrequent with small samples, the estimates given by the method of moments are outside of the parameter space; it does not make sense to rely on them then. That problem never arises in the method of maximum likelihood. Also, estimates by the method of moments are not necessarily sufficient statistics, i.e., they sometimes fail to take into account all relevant information in the sample. The estimators obtained by the method of moments are not as efficient as method of maximum likelihood

7. Is moment estimate uniquely determined? Explain with an example.
Answer:
Let x1, x2… xn be a sample taken from a Normal population with mean 0 and variance σ^2

Then $\mu_1^{'}$ = E(x) = 0 and $\mu_2 = \sigma^2$ since $\mu_2 = \sigma^2 \Rightarrow \mu_2 = \mu_2^{'} - (\mu_1^1)^2 = \mu_2^{'}$ since mean $\mu_1^{'}$ = 0

Now from the method of moments unknown parameter μ and σ^2 are estimated such that $\mu_1^{'}$ = $m_1^{'}$ and $\mu_2^{'}$ = $m_{2'}$

By equating $\mu_2^{'}$ = $m_{2'}$ we get $\sigma^2 = \frac{\sum x_i^2}{n}$

By equating μ_2 = m_2 we get $\hat{\sigma}^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2 = s^2$

Since $m_2 = m_2^{'} - (m_1^{'})^2 \Leftarrow = \frac{\sum x_i^2}{n} - (\bar{x})^2$

If xi~N(0,σ^2) then the moment estimators of σ^2 are

$\hat{\sigma}^2 = \frac{\sum x_i^2}{n}$ and 2) $\hat{\sigma}^2 = s^2$

Hence Moment estimators are not unique

8. Find the moment estimate of θ of the geometric distribution with parameter θ

Answer:

The probability mass function of a geometric variate x is $F(x, \theta) = \theta (1-\theta)^x$

The first population moment $\mu_1^{'}$ = E(x) = (1- θ)/ θ

The first sample moment m1' = $\bar{x}$

Then $\mu_1^{'} = \bar{x} \Rightarrow \frac{1-\theta}{\theta} = \bar{x} \Rightarrow \frac{1}{\theta} - 1 = \bar{x} \Rightarrow \theta = \frac{1}{1+\bar{x}}$

Hence the moment estimator of the parameter θ is 1/ (1+$\bar{x}$)

9. Find the moment estimator of θ in the Beta population of second kind with parameters 1 and θ

Answer:

$x_i \sim \beta_2(1, \theta)$

$$f(xi,\theta) = \frac{1}{\beta(1,\theta)} \frac{x^{1-1}}{(1+x)^{\theta+1}} = \frac{1}{\beta(1,\theta)} \frac{1}{(1+x)^{\theta+1}}$$

When x_i ~ β_2 (m, n)

F (xi, m, n) = $\frac{1}{\beta(m,n)}\frac{x^{m-1}}{(1+x)^{m+n}}$,0<x< ∞

$$\mu_r^{'} = E(x^r) = \int x^r \ \frac{1}{\beta(m,n)}\frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$=\frac{1}{\beta(m,n)}\int\frac{x^{m+r-1}}{(1+x)^{m+n}}dx = \frac{1}{\beta(m,n)}\int\frac{x^{m+r-1}}{(1+x)^{m+r+n-r}}dx = \frac{1}{\beta(m,n)}\beta(m+r,n-r)$$

$$=\frac{\Gamma(m+n)}{\Gamma(n)\Gamma(m)}\frac{\Gamma(m+r)\Gamma(n-r)}{\Gamma(m+n)} = \frac{1}{\Gamma(n)\Gamma(m)}\frac{\Gamma(m+r)\Gamma(n-r)}{}$$

Putting r=1,m=1,n=θ

We get

$$\mu_1^{'} = \frac{\Gamma(1)\Gamma(\theta-1)}{\Gamma(1)\Gamma(\theta)} = \frac{1}{\theta-1}$$

$$\mu_1^{'} = m_1^{'} = \bar{x} \Rightarrow \frac{1}{\theta-1} = \bar{x} \Rightarrow \theta = \frac{1+\bar{x}}{\bar{x}}$$

Hence the moment estimator of θ is $\hat{\theta} = \frac{1+\bar{x}}{\bar{x}}$

10. Find the moment estimate of Θ in the following function whose p.d.f is

F (x,Θ) = $\frac{\theta(\theta+1)x^{\theta-1}}{(1+x)^{\theta+2}}, 0 < x < \infty$

Answer:
We can find the estimate of Θ using the moment

$$\mu_1^{'} = \int xf(x,\theta)dx = \int x\frac{\theta(\theta+1)x^{\theta-1}}{(1+x)^{\theta+2}}dx = \theta(\theta+1)\int\frac{x^{\theta}}{(1+x)^{\theta+2}}dx = \theta(\theta+1)\beta(\theta+1,1)$$

$$=\theta(\theta+1)\,\frac{\Gamma(\theta+1)\Gamma 1}{\Gamma(\theta+2)}=\frac{\theta(\theta+1)\theta\Gamma\theta}{(\theta+1)\theta\Gamma\theta}=\theta$$

Hence a moment estimate of θ is obtained as $\mu_1^{'}=m_1^{'}=\bar{x}$

Therefore Moment estimator of θ is given by $\hat{\theta}=\bar{x}$

11. For the following distribution estimate the parameters α and Θ by the method of moments $f(x, \theta) = \frac{1}{\alpha!\theta^{\alpha+1}} x^{\alpha} e^{-x/\theta},\ 0 < x < \infty$

Answer:
We have to estimate α and Θ based on the sample observations $x_1, x_2, \ldots, x_n$. Hence we need to find $\mu_1^{'}$ and $\mu_2^{'}$

$$\mu_1^{'}=\int xf(x,\theta)dx=\int x\frac{1}{\alpha!\theta^{\alpha+1}}x^{\alpha}e^{-x/\theta}dx=\frac{1}{\alpha!\theta^{\alpha+1}}\int x^{\alpha+1}e^{-x/\theta}dx$$

$$=\frac{1}{\alpha!\theta^{\alpha+1}}\frac{\Gamma(\alpha+2)}{(1/\theta)^{\alpha+2}}=(\alpha+1)\theta---(1)$$

$$\mu_2^{'}=\int x^2 f(x,\theta)dx=\int x^2\frac{1}{\alpha!\theta^{\alpha+1}}x^{\alpha}e^{-x/\theta}dx=\frac{1}{\alpha!\theta^{\alpha+1}}\int x^{\alpha+2}e^{-x/\theta}dx$$

$$=\frac{1}{\alpha!\theta^{\alpha+1}}\frac{\Gamma(\alpha+3)}{(1/\theta)^{\alpha+3}}=(\alpha+1)(\alpha+2)\theta^2---(2)$$

Equation (2) by equation (1)

$$\Rightarrow\frac{\mu_2^{'}}{\mu_1^{'}}=(\alpha+2)\theta---(3)$$

Equation (3) – equation (1) $\Rightarrow\frac{\mu_2^{'}}{\mu_1^{'}}-\mu_1^{'}=(\alpha+2)\theta-(\alpha+1)\theta=\theta$

$$\Rightarrow\theta=\frac{\mu_2^{'}-(\mu_1^{'})^2}{\mu_1^{'}}$$

Substituting this Θ in equation (1)

$\mu_1^{'}$ = (α+1) [$\frac{\mu_2^{'} - (\mu_1^{'})^2}{\mu_1^{'}}$] ⇒ α= $\frac{(\mu_1^{'})^2}{\mu_2^{'} - (\mu_1^{'})^2} - 1$

By the method of moments $\mu_1^{'}$ = $m_1^{'}$ and $\mu_2^{'}$ = $m_{2'}$

$$\hat{\theta} = \frac{\hat{\mu}_2^{'} - (\hat{\mu}_1^{'})^2}{\hat{\mu}_1^{'}} = \frac{m_2^{'} - (m_1^{'})^2}{m_1^{'}} = \frac{(\sum xi^2 / n) - \bar{x}^2}{\bar{x}}$$

$$\hat{\alpha} = \frac{(\hat{\mu}_1^{'})^2}{\hat{\mu}_2^{'} - (\hat{\mu}_1^{'})^2} - 1 = \frac{(m_1^{'})^2}{(m_2^{'}) \ - ((m_1^{'})^2} - 1 = \frac{\bar{x}^2}{(\sum xi^2 / n) - (\bar{x}^2)} - 1$$

12. Estimate the parameter p in sampling from a Binomial population with parameter n unknown by the method of moments

Answer:

Since only one parameter is unknown we need to find $\mu_1^{'}$ only. But for a binomial variable with parameters n and p

$\mu_1^{'}$ = E(x) = np

Now from the method of moments unknown parameter p is estimated such that $\mu_1^{'}$ = $m_1^{'}$

But $m_1^{'}$ = $\frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$

Hence $\mu_1^{'} = np = m_1^{'} = \bar{x}$

Therefore Moment estimator of p is given by $\hat{p}$ = $m_1^{'}$ /n= $\frac{\bar{x}}{n}$

13. Obtain an moment estimator of θ of a Uniform population with parameters 0 and θ

Answer:

We know that when $x_i \sim U(0, \theta)$

The p.d.f f(x) = 1/θ; 0<xi<θ

The first population moment $\mu_1^{'} = \theta / 2$

Now from the method of moments unknown parameter θ is estimated such that $\mu_1^{'}$ = $m_1^{'}$

But m$_1{}'$ = $\frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$

Hence $\mu_1' = \theta/2 = m_1' = \bar{x}$

Therefore Moment estimator of θ is given by $\hat{\theta} = 2\bar{x}$

14. Obtain the moment estimators of mean and variance of a Normal population with parameters μ and σ^2

Answer:

X i~N (μ ,σ^2)

Since two parameters are unknown we need to find μ_1' and μ_2' . But for a Normal variate with parameters μ and σ^2

μ_1' = E(x) = μ and $\mu_2 = \sigma^2$

Now from the method of moments unknown parameter μ and σ^2 are estimated such that μ_1' = m$_1{}'$ and μ_2' = m$_{2'}$

But m$_1{}'$ = $\frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$ and m$_2{}'$ = $\frac{\sum_{i=1}^{n} x_i^{\,2}}{n}$

Hence $\mu_1' = \mu = m_1' = \bar{x}$

From the method of moments unknown parameter μ is estimated such that μ_1' = m$_1{}'$

Hence $\hat{\mu} = \bar{x}$

But since $\mu_2 = \sigma^2 \Rightarrow \mu_2 = \mu_2{}' - (\mu_1^1)^2 = \sigma^2$

Hence an estimate of variance

$$\Rightarrow \hat{\mu}_2 = \hat{\sigma}^2 = m_2' - (m_1^1)^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2 = s^2$$

Therefore Moment estimator of parameter μ and σ^2 are give as

$\hat{\mu} = \bar{x}$ and $\hat{\sigma}^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2 = s^2$

15. Let x1,x2,….,xn be a sample taken from a Normal population with mean 0 and variance θ. Find the moment estimate of θ

Answer:

xi~N(0,θ)

The first population moment μ_1' = E(x) = 0

The second population moment μ_2'= E(x^2)=V(x)+$[E(x)]^2$

= θ+0 = θ

μ_2'=θ

Now by the method of moments unknown parameter θ is estimated such that μ_1' = m_1' and μ_2' = m_2'

But m_1' = $\frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$ and m_2' = $\frac{\sum_{i=1}^{n} x_i^2}{n}$

But s^2 = $\frac{1}{n}\sum_{i=1}^{n}(xi - \bar{x})^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2 \Rightarrow s^2 + (\bar{x})^2 = \frac{\sum x_i^2}{n}$

Hence $\mu_1' = \mu = m_1' = \bar{x}$=0 --- (1)

$$\mu_2' = m_2' \Rightarrow \theta = s^2 + (\bar{x})^2 = \frac{\sum x_i^2}{n}$$

From (1)

Therefore Moment estimator of parameter θ is

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2}{n}$$