Frequently Asked Questions

1. State Linderberg Central Limit Theorem.

Answer:

We assume that $X_1, \ldots\ldots, X_n$ are independent random variables with means μi and respective variances σ_i^2. then

For every ε > 0

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{i=1}^{n} \mathbb{E}\left[(X_i - \mu_i)^2 \cdot \mathbf{1}_{\{|X_i-\mu_i|>\varepsilon s_n\}}\right] = 0$$

$$\lim_{n\to\infty} \frac{1}{sn^2} \sum_{i=1}^{n} E\left[(X_i - \mu_i)^2 . I_{\{|Xi-\mu_i|>\varepsilon s_n\}}\right] = 0$$

where $\mathbf{1}_{\{\ldots\}}$ is the indicator function. Then the distribution of the standardized sums $\frac{1}{s_n}\sum_{i=1}^{n}(X_i - \mu_i)$ converges towards the standard normal distribution N(0,1) where sn,the standard deviation of Sn

2. When do we say that a random variable is asymptotically Normally distributed?

Answer:

If the distribution of a random variable Y depends on a parameter n and if there exists two quantities μ or σ (which may or may not depend on n such that

$$\lim_{n\to\infty} P\left[\frac{Y-\mu}{\sigma} \le t\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp(-x^2/2)dx$$

Then we say that Y is asymptotically Normally distributed with mean μ and variance σ^2

3. State the significant contribution of Central Limit Theorem to measurement errors.

Answer:

Not only does the CLT give us the method for approximating the distribution of the sum of the R.Vs but also it helps to obtain remarkable facts that the empherical frequencies of so many naturally occurring populations exhibit a bell shaped (that is normal) curve.

Indeed, one of the first uses of the CLT was to provide theoretical justification of the empherical fact that the measurement errors tend to be Normally distributed. That is by regarding an error in measurement as being composed of the sum of a large number of small independent errors, the CLT implies that it should be approximately Normal.

4. How do you apply CLT to measurement errors?

Answer:

For instance an error of measurement in astronomy can be regarded as being equal to the sum of the small errors caused by such things as

1) Temperature effects on the measuring devise
2) Bending of the devise caused by the rays of the sun
3) Elastic effects
4) Air currents
5) Air vibrations
6) Human errors

Therefore by CLT the total measurement errors will approximately follow Normal distribution. From this it follows that a histogram of errors resulting from a series of measurements of the same object will tend to follow a bell-shaped Normal curve.

5. Distinguish between Law of Large Numbers and CLT.

Answer:

By the law of large numbers, the sample averages converge in probability and almost surely to the expected value μ as n tends to infinity. The classical central limit theorem describes the size and the distributional form of the stochastic fluctuations around the deterministic number μ during this convergence.

6. What is the significance of Linderberg theorems?

Answer:

The seeds of the Central Limit Theorem, or CLT, lie in the work of Abraham de Moivre, who, in 1733, not being able to secure himself an academic appointment, supported himself consulting on problems of probability, and gambling. He approximated the limiting probabilities of the Binomial distribution, the one which governs the behavior of the number Sn of success in an experiment which consists of n independent trials, each one having the same probability $p \in (0, 1)$ of success.

Sufficiency is proved by Lindeberg in 1922 and necessity by Feller in 1935. Lindeberg-Feller CLT is one of the most far-reaching results in probability theory. Nearly all generalizations of various types of central limit theorems spin from Lindeberg-Feller CLT, such as, for example,CLT for martingales, for renewal proceses, or for weakly dependent processes. The insights of the Lindeberg condition are that the wild values of the random variables, compared with sn,the standard deviation of Sn as the normalizing constant, are insignificant and can be truncated off without affecting the general behavior of the partial sum Sn.

7. State Linderberg – Levy Central Limit Theorem.

Answer:

Let {Xk} be the sequence of independent and identically distributed random variables with with $E(X_k)=\mu$ and $V(X_k) = \sigma^2$ which is finite , then

$\sum_{k=1}^{n} \frac{(Xi-\mu)}{\sigma\sqrt{n}}$ is asymptotically Normal with mean 0 and variance 1

Or in other words as n approaches infinity, the random variables √n($\overline{Xn}$ − μ) converge in distribution to a Normal , that is $N(0, \sigma^2)$

$$\sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \mu\right) \xrightarrow{d} \mathcal{N}(0,\ \sigma^2).$$

8. State Lyapunov's CLT.

Answer:

Let {Xk} be the sequence of independent random variables with with $E(X_k)=\mu_k$ and $V(X_k) = \sigma_k^2$ and $E|Xk-\mu k|^{2+\partial} < \infty$ then

$\sum_{k=1}^{n} \frac{(X_k-\mu_k)}{C_n}$ is asymptotically Normal with mean 0 and variance 1 provided

$\lim_{n\to\infty}\left[\frac{1}{C_n^{2+\partial}}\sum_{k=1}^{n} E|Xk-\mu k|^{2+\partial}\right] = 0$ for some $\partial, 0 < \partial \le 1,$ where $C_n^2 = \sum_{k=1}^{n}\sigma_k^2$

9. State Linderberg – Feller Central Limit Theorem.

Answer:

Let {Xk} be the sequence of independent random variables with with $E(X_k)=\mu_k$ and $V(X_k) = \sigma_k^2$ which is finite and Fk be the distribution function of X k then

i) $\sum_{k=1}^{n} \frac{(X_k-\mu_k)}{C_n}$ is asymptotically Normal with mean 0 and variance 1 and

ii) $\lim_{n\to\infty}\max_{k\le n}\frac{\sigma_k}{C_n} = 0$ holds if and only if for all $\varepsilon > 0$

$$b_n(\varepsilon) = \left[\frac{1}{C_n^2}\sum_{k=1}^{n}\int_{|x-\mu_k|\ge\varepsilon C_n}(x-\mu_k)^2\,dF_k(x)\right] \to 0 \text{ as } n\to\infty$$

In the above statement $C_n^2 = \sum_{k=1}^{n}\sigma_k^2$

10. What is Linderberg –Feller CLT in terms of distribution function of Normal distribution.

Answer:

If the random variates X_1, X_2, ... satisfy the Lindeberg condition, then for all a < b ,

$$\lim_{n\to\infty} P\left(a < \frac{S_n}{s_n} < b\right) = \Phi(b) - \Phi(a),$$

where Φ is the normal distribution function.

11. How can we say that Linderberg- Levy Theorem is the simple corollary to the Linderberg –Feller Theorem?

Answer:

Linderberg- Levy Theorem also follow as simple corollaries to the Linderberg –Feller Theorem

For the case of i.i.d random variables with finite variance σ^2 condition (1) is always satisfied for

$$b_n(\varepsilon) = \left[\frac{1}{n\sigma^2}\sum_{k=1}^{n} \int_{(x-\mu)^2 \geq \varepsilon^2 n\sigma^2} (x-\mu)^2 dF(x)\right] = \frac{1}{\sigma^2}\int_{y^2 \geq \varepsilon^2 n\sigma^2} y^2 dF(y) \to 0$$

as $n \to \infty$ for every $\varepsilon > 0$ because $n\sigma^2 \to \infty$

12. What are the general applications of these Theorems?

Answer:

If the X_k's are Gaussian S is Gaussian and as $n\to\infty$, S is again Gaussian.

If the X_k's are Binomial , S is Binomial and as $n\to\infty$, S is Gaussian.

If the X_k's are Poisson, S is Poisson. and as $n\to\infty$, S is Gaussian.

If the X_k's are Gamma, S is Gamma and as $n\to\infty$, S is Gaussian.

If the X_k's are Negative Binomial, S is Negative Binomial and as $n\to\infty$, S is Gaussian.

But CLT is not applied for Cauchy distribution

If the X_k's are Cauchy, S is Cauchy but as $n\to\infty$, S is still Cauchy.

13. State the assumptions behind Linderberg-Levy CLT?

Answer:

The basic assumptions behind Linderberg Levy CLT are

i) The variables are i.i.d
ii) Variances of Sn= V(X1+X2+…+Xn) should exist or E(Xi^2) should exist for all i=1,2,..,n

14. If Yn is a Binomial variate with parameter n and p then prove that $\lim_{n\to\infty} P\left(a < \frac{Yn - np}{\sqrt{npq}} < b\right) = \Phi(b) - \Phi(a)$

Answer:

Let X1,X2,...,Xk be i.i.d Bernoulli variate that is Xi~B(1,p). Then Sn= X1+X2+…+Xn ~B(n,p)

By Linderberg Feller CLT $\lim_{n\to\infty} P\left(a < \frac{Sn - E(Sn)}{s_n} < b\right) = \Phi(b) - \Phi(a)$

But given Yn ~B(n,p). Then E(Yn)=np and V(Yn)=npq. Hence using Yn instead of Sn

$$\lim_{n\to\infty} P\left(a < \frac{Yn - np}{\sqrt{npq}} < b\right) = \Phi(b) - \Phi(a)$$

15. If Yn is distributed as Poisson with parameter n then prove that $\lim_{n\to\infty} P\left(a < \frac{Yn - n}{\sqrt{n}} < b\right) = \Phi(b) - \Phi(a)$ and $\lim_{n\to\infty} P(Yn \le n) = \frac{1}{2}$

Answer:

Let X1,X2,...,Xk be i.i.d Poisson variates with parameter 1then

Sn= X1+X2+…+Xn ~P(n) which implies Yn=Sn. Then

By Linderberg Feller CLT

$$\lim_{n\to\infty} P\left(a < \frac{Yn - n}{\sqrt{n}} < b\right) = P\left(a < \frac{Sn - n}{\sqrt{n}} < b\right) = \Phi(b) - \Phi(a)$$

In particular take a=$-\infty$ and b =0

$$\lim_{n\to\infty} P\left(a < \frac{Yn - n}{\sqrt{n}} < b\right) = P\left(\frac{Yn - n}{\sqrt{n}} < 0\right) = P(Yn < 0) - - - - (1)$$

Also $\Phi(b)-\Phi(a)=\Phi(0)-\Phi(-\infty)=0.5-0=0.5------(2)$

From equations (1) and (2)

$$\lim_{n\to\infty} P(Yn\leq n)=\frac{1}{2}$$