Frequently Asked Questions

1. What do you mean by Law of Large Numbers?

Answer:

In probability theory, the **law of large numbers** (**LLN**) is a theorem that describes the result of performing the same experiment a large number of times. According to the law, the average of the results obtained from a large number of trials should be close to the expected value, and will tend to become closer as more trials are performed.

2. Illustrate Law of Large Numbers with an example.

Answer:

A single roll of a six-sided die produces one of the numbers 1, 2, 3, 4, 5, or 6, each with equal probability. Therefore, the expected value of a single die roll is

$$\frac{1+2+3+4+5+6}{6} = 3.5.$$

According to the law of large numbers, if a large number of six-sided dice are rolled, the average of their values (sometimes called the sample mean) is likely to be close to 3.5, with the accuracy increasing as more dice are rolled.

It follows from the law of large numbers that the empirical probability of success in a series of Bernoulli trials will converge to the theoretical probability. For a Bernoulli random variable, the expected value is the theoretical probability of success, and the average of n such variables (assuming they are independent and identically distributed (i.i.d.)) is precisely the relative frequency.

3. Why a Law of Large Numbers is considered important?

Answer:

The LLN is important because it "guarantees" stable long-term results for random events. For example, while a casino may lose money in a single spin of the roulette wheel, its earnings will tend towards a predictable percentage over a large number of spins. Any winning streak by a player will eventually be overcome by the parameters of the game.

In practice, we are interested in estimating parameters for which we use statistics. The law of Large Numbers will tell us when a statistic (itself a random variable) will converge to the parameter being estimated in some sense .

It is also important to remember that the LLN only applies (as the name indicates) when a *large number* of observations are considered. There is no principle that a small number of observations will converge to the expected value or that a streak of one value will immediately be "balanced" by the others.

4. Narrate the history behind Weak Law of Large Numbers.

Answer:

A theorem of importance usually known as Bernoulli's theorem was first published posthumously in the first part of the 18^{th} century in Jacobi Bernoulli's famous book Ars conjectandi. French mathematician Simeon Poisson gave it the name "Law of Large Numbers". According to Bernoulli, it took him 20 years to complete the theorem.

Later Poisson proved an analogous theorem at the beginning of the 19^{th} century under more general conditions. The Russian mathematician Chebyshev's discovered his proof in 1866 using this inequality. Later Markov obtained a more general result using Tchebyscheff's reasoning.

In 1928, Khinchin showed that for a sequence of independent and identically distributed random variables the Law of Large numbers holds if the expectation exists. The above works were concerned with what is known as Weak Law of Large Numbers and gave rise to the theorem of Weak Law of Large Numbers

5. State Weak Law of Large Numbers.

Answer:

In words, WLLN states that if a trial is reproduced a large number of times n, then it becomes exceedingly improbable that the average of the outcomes of these n trials will differ significantly from the expected value of one outcome as n grows without limit. In more technical terms, the Weak Law of Large Numbers states that:

Suppose $\{X_i\}$ is an infinite sequence of i.i.d. random variables with common mean μ, If we define Y_n as the r.v. equal to the mean of the first n X_is, Then, for any ε, the probability for a realization of Y_n to fall more than ε away from μ tends to 0 as n grows without limit.

No matter how small ε, all you have to do ,to make the probability for the mean of the first n terms to differ from the mean μ by more than ε to be as small as you wish, is to make n large enough.

In the vocabulary of Estimation, the WLLN states that the sample mean is a consistent estimator of the population mean.

6. Distinguish between Weak Law of Large Numbers and Strong Law of Large Numbers.

Answer:

The difference between the strong and the weak version is concerned with the mode of convergence being asserted Already we are familiar with two types of convergence of random variables and its interpretations namely Convergence in distribution and Convergence in probability

The *weak law* states that for a specified large n, the average $\overline{X}_n$ is likely to be near μ. Thus, it leaves open the possibility that $|\overline{X}_n - \mu| > \varepsilon$ happens an infinite number of times, although at infrequent intervals.

The *strong law* shows that this almost surely will not occur. In particular, it implies that with probability 1, we have that for any $\varepsilon > 0$ the inequality $|\overline{X}_n - \mu| < \varepsilon$ holds for all large enough n.

The Laws of Large Numbers make statements about the convergence of $\overline{Xn}$ to μ. Both laws relate bounds on sample size, accuracy of approximation, and degree of condense. The Weak Laws deal with limits of probabilities involving $\overline{Xn}$. The Strong Laws deal with probabilities involving limits of $\overline{Xn}$.

7. Illustrate Weak Law of Large Numbers with an example.

Answer:

The coin-tossing paradigm is convenient example for the application of WLLN.
Consider A fair coin and, for a given *n*, the set of all the possible outcomes of a sequence of *n* tosses of this coin. There are exactly 2^n such sequences, and because of the fairness of the coin, they all have the same probability 2^{-n} of occurring when an actual series of *n* tosses is performed.
Suppose these sequences are numbered 1, 2, ..., 2^n in an arbitrary way. For sequence Number *i*, denote μ_i the average number of Heads per toss in the sequence: $\mu_i = 1/n$. Number of Heads in the sequence
Then, take an arbitrary small positive real number *ε*, and count the number *N*(*n*, *ε*) of sequences such that μ_i departs from 1/2 by more than *ε*. We can show that the proportion of these "deviant" sequences: $N(n, \varepsilon) / 2^{n\backslash}$ tends to 0 as *n* grows without limit. Therefore, the WLLN can easily be derived directly in the case of a fair coin tossing

The WLLN tells us that that ultimately, the numbers of Heads and Tails must be about equal. After this incredibly unlucky opening sequence, the following tosses must therefore produce mostly Tails for the game to return to a roughly balanced count of Heads and Tails. Hence, we may expect the following tosses to generate mostly Tails.

In other words, an excess of Heads in an opening sequence must cause an increase of the probability of Tails for the ensuing tosses.

8. State Tchebyscheff's Theorem of WLLN.

Answer:

Tchebyscheff's Theorem of WLLN

Let {Xn} be a sequence of independent random variables such that $E(Xi)=\mu_i$ and $V(Xi) = \sigma_i^2$.

Let Bn= V(X1+X2+….+Xn) is finite

Then

$$P\{\left|\frac{|X_1 + X_2 + ... + X_n}{n} - \frac{\mu_1 + \mu_2 + ... + \mu_n}{n}\right| < \varepsilon\} \geq 1 - \eta$$ **for all** $n > n_0$, **where** ε

and η are arbitrary small positive numbers $\Rightarrow \overline{Xn} - \overline{\mu} \xrightarrow{P} 0$

provided $$\lim_{n \to \infty} \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 = \lim_{n \to \infty} \frac{B_n}{n^2} = 0$$

Where $\overline{Xn} = \frac{\sum_{i=1}^{n} Xi}{n}$ and $\overline{\mu} = \frac{\sum_{i=1}^{n} \mu i}{n}$

9. What are the conditions to be assumed for WLLN?

Answer:

For the existence of the law, we assume the following conditions:

i) E(Xi) exists for all i

ii) Bn= V(X1+X2+….+Xn) exists and

iii) $\frac{B_n}{n^2} \rightarrow 0$ as $n \rightarrow \infty$

Condition (i) is necessary without it the law cannot be stated. But the conditions (ii) and (iii) are not necessary (iii) is however a sufficient condition.

10. State the WLLN for independently identically distributed Random variables.

Answer:

If the variables X1,X2,…, Xn are independently and identically distributed , then E(Xi)=μ and V(Xi) = σ^2 . Bn= n σ^2 , $\frac{B_n}{n^2} = \frac{n\sigma^2}{n^2} \rightarrow 0$

as $n \rightarrow \infty$

If σ_i^2 =σ^2 then the condition of the theorem is automatically satisfied and we have the result $\overline{Xn} - \mu \xrightarrow{P} 0$. If further the random variables {Xi} are independent and identically distributed (i.i.d) with E(Xi) = μ and V(Xi) = σ^2 then $\overline{X}n \xrightarrow{P} \mu$ that is the sample mean converges stochastically to the population mean.

11. How do you interpret WLLN?

Answer:

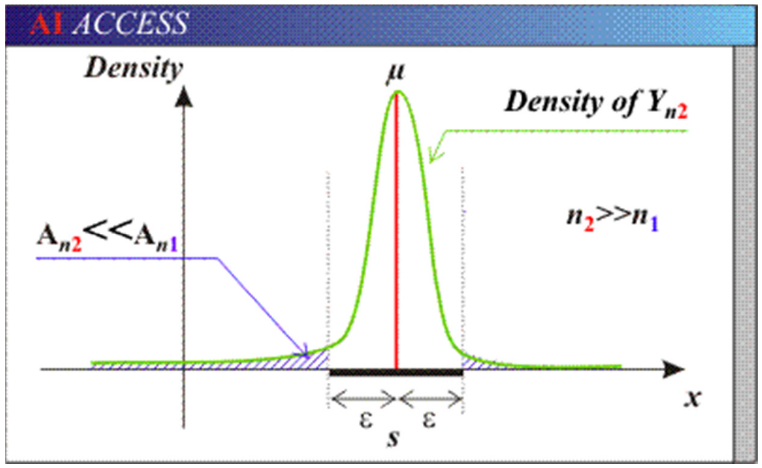
Interpreting the theorem of WLLN, the weak law essentially states that for any nonzero margin specified, no matter how small, with a sufficiently large sample there will be a very high probability that the average of the observations will be close to the expected value, that is, within the margin.

Convergence in probability is also called weak convergence of random variables.

12. Highlight the graphic interpretation of the Weak Law of Large Numbers.

Answer:

The WLLN receives a simple graphic interpretation. For each value of n, the random variable Y_n has a probability distribution (that we here assume to be continuous) which is represented by the green curve in the illustration below. The area under the curve is always 1.
Now position a 2ε long segment ***s*** on top of μ. Denote A_n the area under the curve outside ***s***.



A_n is the probability for Y_n to be different from μ by more than ε (in absolute value).
The WLLN states that, for a given ε, A_n tends to 0 as n grows without limit. In other words, outside of **s**, the "tails" of the distribution of Y_n become negligible when n tends to infinity (lower image of the above illustration).

13. Briefly explain the limitations of WLLN.

Answer:

Mean and variance

The above expressions make an explicit reference to the common **mean** μ of the variables.
In addition, the WLLN will appear to be a consequence of Chebyshev inequality, which requires the variables to also have a **variance**.
Therefore, the WLLN **seems** to apply only to variables that have at least a mean **and** a variance. In fact, the existence of the variance is **not** required.

The existence of the mean is of course always required. Therefore, for example, the WLLN does not apply to samples drawn from the Cauchy distribution, as this distribution has no mean. We already noticed that the distribution of the mean of a sample drawn from a Cauchy distribution does not depend on the sample size. Therefore, there is no "shrinking" of this distribution as the sample size increases.

Independence

We stated that the WLLN applies to **independent** or i.i.d random variables. Assuming the independence of variables is so common in Statistics that we sometimes forget how strong a restriction this is.
There are some counter-example that deals with variables that are indeed identically distributed but that are not independent. A consequence of the breakdown of the independence hypothesis will be that the WLLN does not apply to this sequence of variables. It expects the variables to be at least independently distributed.

14. Why WLLN is considered weak?

Answer:

The term "Weak" refers to the way the sample mean converges to the distribution mean. At first sight, it may seem that there is no better way to converge than what we described here, and which is known as "convergence in probability".
However, it turns out that the sample mean converges to the distribution mean in a much "stronger" way than just "in probability". This convergence is almost sure, and the Weak Law of Large Numbers is in fact superseded by a "Strong" Law of Large Numbers.

15. Let Xi assume the values I and –I with equal probabilities. Show that the law of large numbers cannot be applied to the independent variables X1,X2,…

Answer:

We have P(Xi=i)=1/2, P(xi=-i) = ½

E(Xi) = 0

$V(Xi) = E(Xi^2) – (E(Xi))^2=i^2$

Bn= V(X1+X2+…+Xn)= V(X1)+V(X2)+…+V(Xn)

$= (1^2 +2^2+\ldots+n^2) = n(n+1)(2n+1)/6$

$$\frac{B_n}{n^2} \to \infty \text{ as } n \to \infty$$

The sufficient condition of WLLN is not satisfied by the given random variables. Hence, Law of large numbers does not hold.