1. Introduction and Properties of Estimators

Welcome to the series of E-learning modules on Practical problems on Estimation by Consistency, Unbiased and Relatively Efficient. In this module we are going cover the conditions required for Unbiasedness, consistency and relative efficiency and the problems to demonstrate the procedure to obtain these estimators.

By the end of this session, you will be able to :

- Describe the properties of estimators
- Understand Consistency, Unbiased and relatively efficient estimators
- Describe the procedure to obtain these estimators

Estimator is accepted or rejected depending on its sampling properties.
A statistic whose distribution concentrates as closely as possible near the true value of the parameter may be regarded as the best estimator.
We expect the estimator to have four desirable properties like consistency, unbiasedness, efficiency and sufficiency to be considered as the best estimator. Let us just recollect the basic conditions required for the estimators to fulfil these properties

Consistency:
A desirable property of a good estimator is that the accuracy should increase when the sample size becomes larger.

An estimator is said to be **consistent** if the variance of its sampling distribution decreases with increasing sample size. This is a good property because it means that if you make the effort to collect data from a larger random sample, you should end up with a more accurate estimate of the population parameter.

Suppose T_n is an estimator of an unknown parameter theta. Tn is said to be consistent for theta if Tn converges to theta in probability as n tends to infinity
That is Probability of modulus of Tn minus theta less than epsilon tends to one as n tends to infinity OR Probability of modulus of Tn minus theta greater than epsilon tends to zero as n tends to infinity

Consistency is a limiting property. Moreover several consistent estimators may exist for the same parameter. For example: In sampling from a Normal population, Normal (theta, sigma square) both the sample mean and the median are the consistent estimators of the population mean theta.

Necessary and sufficient condition for consistency
Suppose Tn is unbiased for g of theta and Variance of Tn tends to zero as n tends to infinity then Tn is consistent for g of theta OR
An estimator Tn is an consistent estimator for g of theta (a function of theta) if Expected value of Tn is equal to g of theta and Variance of Tn tends to zero as n tends to infinity

2. Properties Contd.

Unbiasedness:
The expected value (mean) of the estimate's sampling distribution is equal to the underlying population parameter; that is, there is no upward or downward bias. Suppose Tn is an estimator of an unknown parameter theta then Tn is said to be unbiased for theta if Expected value of Tn is equal to theta

Tn is said to be asymptotically unbiased for theta if Expected value of Tn is equal to theta as n tends to infinity.
If Expected value of Tn is not equal to theta then Tn is said to be biased for theta. Bias in estimation is B of theta is equal to Expected value of Tn minus theta
For example:
The sample mean x bar is an unbiased estimator of the population mean θ
Expected value of x bar is equal to theta

But, the sample variance s square equal to summation, I ranging from one to n, (xi minus x bar) whole square by n is a biased estimator of the population variance sigma square
An unbiased estimator of the population variance sigma square is given by s square is equal to summation (xi minus x bar) whole square by n minus one i ranging from one to n.

Efficiency –
It is possible that there may be several consistent estimators for the same population parameter. For example, In case of Normal population sample mean and the sample median are both consistent estimators of the population mean, Thus it is necessary to have a criterion to decide a better estimator within the class of consistent estimators.

While there are many consistent estimates of the same parameter, the most efficient has a sampling distribution with the smallest variance. Of two consistent estimators for the same parameters, the statistic with the small sample variance is said to be ‘more efficient ‘.

Thus if t and t dash are both consistent estimators of theta and Variance of t is less than variance of t dash then t is said to be more efficient than t dash in estimating theta. If a consistent estimator exists whose sampling variance is less than that of any other consistent estimator it is said to be most efficient.

Further a relative efficiency of T one with respect to T two is defined as
E of T one , T two is equal to Variance of T two by variance of T one

a) If E of (T one), (T two) is equal to one then the both (T one) and (T two) are equally efficient
b) If E of (T one) , (T two) is greater than one, then (T one) is more efficient than (T two)
c) If E of (T one) , (T two) is less than one then (T two) is more efficient than (T one)

3. Problems Pertaining to Estimation of Estimators

Now let us discuss about certain problems related to the estimation of consistent, unbiased and efficient estimators

Problem 1:

Show that when sampling from a Normal population, a sample variance ,

S square is equal to one by n summation (xi minus x bar) whole square, is biased for the population variance but is asymptotically unbiased.

Also find the unbiased estimator of the population variance and hence obtain an unbiased estimate of the population variance sigma square from the following sample of size eight from a Normal population.

Seventy four point one, seventy seven point two, seventy four point four, seventy four, seventy three point eight, seventy nine point three, seventy five point eight, eighty two point eight .

Solution:

Given

S square is equal to one by n summation (xi minus x bar) whole square. We know that ,

n into s square by sigma square follows Chi square with (n minus one) degrees of freedom.

Which implies Expected value of (n into s square by sigma square) is equal to n minus one which implies expected value of s square is equal to (n minus one) by n into sigma square which is not equal to sigma square

Therefore ‘s’ square is biased for sigma square.

But limit as n tends to infinity of Expected value of s square is equal to limit as n tends to infinity of (n minus one by n) into sigma square which tends to sigma square.

Therefore s square is asymptotically unbiased for sigma square.

Expected value of s square is equal to (n minus one by n) into sigma square implies Expected value of n into s square by (n minus one) is equal to sigma square.

Therefore n into s square by (n minus one) is an unbiased estimator of sigma square.

A sample variance s square is equal to one by n summation (xi minus x bar) whole square which is equal to one by n into (summation xi square minus n into x bar square).

Which is equal to one by eight into (forty six thousand seven hundred and ninety eight point four two minus eight into (seventy six point four two five square) which is equal to nine point zero two. Therefore n into s square by n minus one is equal to eight into nine point zero two by seven is equal to ten point three one is an unbiased estimate of the population variance.

4. Problem 2

Problem 2:
Suppose X and Y are independent random variables with the same unknown mean mu. Both X and Y have variance as thirty six. Let T is equal to a X plus b Y be an estimator of mu

i) Show that T is an unbiased estimator of mu if a plus b is equal to one
ii) If ‘a’ is equal to one by three and b is equal to two by three what is the variance of T?
iii) If ‘a’ is equal to one by two and b is equal to one by two what is the variance of T?
iv) What choices of ‘a’ and ‘b’ minimizes the variances of T subject to the requirement that T is the unbiased estimator of mu?

Solution:
X and Y are independent random variables with the same unknown mean mu. Then Expected value of (X) is equal to Expected value of (Y) equal to mu
Both X and Y have variance as thirty six. Then Variance of (X) is equal to Variance of (Y) is equal to thirty six.
Given T is equal to ‘a’ X plus b Y .

i) T is an unbiased estimator of mu if and only if
Expected value of (T) is equal to mu which implies Expected value of (a X plus b Y) is equal to mu which is equal to ‘a’ into Expected value of X plus b into expected value of Y equal to ‘a’ into mu plus b into mu which is equal to (a plus b) into mu which is equal to mu
Left hand side will be equal to Right Hand Side if ‘a’ plus b is equal to one
Hence T is an unbiased estimator of mu if ‘a’ plus b is equal to one

ii)Variance of (T) is equal to Variance of (a X plus b Y) is equal to ‘a’ square variance of X plus b square variance of Y which is equal to a square into thirty six plus b square into thirty six which is equal to thirty six into (a square plus b square)
If a is equal to one by three and b is equal to two by three then Variance of T is equal to thirty six into (one by nine plus four by nine) which is equal to twenty.

iii)If a is equal to one by two and b is equal to one by two then Variance of (T) is equal to thirty six into (one by four plus one by four) which is equal to eighteen

iv) T is unbiased for μ if Expected value of (T) is equal to mu.
From (one)
T is an unbiased estimator of mu if ‘a’ plus b is equal to one. When ‘a’ is equal to one by three and b is equal to two by three , ‘a’ plus b is equal to one But Variance of (T) is equal to twenty.
When ‘a’ is equal to one by two and b is equal to one by two then ‘a’ plus b is equal to one. But Variance of T is equal to eighteen.

Hence Variance of (T) is minimum for ‘a’ is equal to one by two and b is equal to one by two . Therefore when ‘a’ is equal to b is equal to one by two, variance of T is minimum subject to the requirement that T is the unbiased estimator of mu.

5. Problem 3 and 4

Problem 3:
If X one, X two, X three is a random sample of size three from a population with mean mu and variance sigma square . T one, T two and T three are the estimators used to estimate the mean value mu where
T one is equal to X one plus X two minus X three,
T two is equal to two X one plus X two minus four X three and
T three is equal to (lambda X one plus X two plus X three) divided by three.
a) Find whether T one and T two are unbiased estimators
b)Find the value of lambda such that T three is an unbiased estimator of mu
c)With this value of lambda is T three a consistent estimator?
d)Which is an efficient estimator?

Solution:
Since X one, X two, X 3 is a random sample of size three from a population with mean mu and variance sigma square Expected value of (x i) is equal to mu and Variance of (X i) is equal to sigma square and Covariance of (X i, X j) is equal to zero for all i not equal to j.
Given T one is equal to X one plus X two minus X three.
T two is equal to two X one plus X two minus four X three.

Expected value of (T one) is equal to Expected value of (X one plus X two minus X three) . This is equal to Expected value of (X one) plus Expected value of (X two) minus Expected value of (X 3) which is equal to mu plus mu minus mu.
Because Expected value of (X i) is equal to mu for all i equal to one,two,three.
Therefore Expected value of (T one) is equal to mu , T one is unbiased for mu.
Expected value of (T two) is equal to Expected value of (two into X one plus three into X two minus four into X three) which is equal to two into Expected value of (X one) plus three into Expected value of (X two) minus four into Expected value of (X three) is equal to two into mu plus three mu minus four mu which is equal to mu. Therefore since Expected value of (T two) is equal to mu. T two is unbiased for mu.
Hence T one and T two are the unbiased estimators of mu.

b) If (T three) is unbiased for mu then Expected value of (T three) is equal to mu.
Expected value of (T three) is equal to mu implies Expected value of(lambda X one plus X two plus X three) by three) is equal to mu.
[lambda into Expected value of X one plus Expected value of X two plus Expected value of X three divided by three is equal to mu.
[lambda mu plus mu plus mu) by three is equal to mu implies lambda mu is equal to mu implies lambda is equal to one

c) With lambda equal to one, T three is equal to (X one plus X two plus X three) divided by three which is nothing but sample mean x bar.
Since sample mean is a consistent estimator of the population mean mu by weak law of

large numbers, T three is a consistent estimator of mu.

d) An efficient estimator is the one which has the least variance
Variance of (T one) is equal to Variance of (X one plus X two minus X 3) which is equal to Variance of (X one) plus Variance of (X two) plus Variance of (X three) which is equal to three into sigma square .
Variance of (T two) is equal to Variance of (two X one) plus (three X two) minus (four X three) which is equal to four into Variance of (X one) plus nine into Variance of (X two) plus sixteen into Variance of (X three) which is equal to twenty nine sigma square
Variance of (T three) is equal to Variance of [(X one)plus (X two) plus (X 3) divided b y three]which is equal to Variance of (X one) plus (X two) plus (X three) divided by nine which is equal to three sigma square divided by nine which is equal to sigma square by three.

Hence we can observe that Variance of (T three) is less than Variance of (T one) is less than Variance of (T two).Therefore (T 3) is an efficient estimator of mu as compared to (T one) and(T two)

Problem 4:
The sample one, three, zero, six, one, seven, two, two, zero, three, one, one drawn from a population with density function f of (x) is equal to one by theta, zero less than x less than theta. Obtain an unbiased estimate of population parameter theta.

Solution:
Given f of (x) is equal to one by theta, zero less than x less than theta.
then X follows Uniform distribution taking values in between zero and theta for which,
Expected value of (X) is equal to theta by two and Variance of (X) is equal to theta square by twelve.
Suppose t is an unbiased estimator of the population mean then Expected value of (t) is equal to population mean and for the given density function population mean is theta by two
If x is unbiased for theta then x bar is also unbiased for theta.

Expected value of (X) is equal to theta by two implies Expected value of two into x is equal to theta or Expected value of two into x bar is equal to theta.
Hence two x bar is an unbiased estimator of the given population's parameter theta .
X bar is equal to summation xi by n which is equal to twenty seven by twelve which is equal to two point two five
Therefore an unbiased estimate of the given population's parameter theta is two into two point two five which is equal to four point five

Here's a summary of our learning in this session:

- Required properties of estimators
- Conditions for consistency and unbiasedness
- Conditions for efficiency and relative efficiency
- Associated practical problems to demonstrate the procedure to obtain these estimators and its values