1. Introduction & Un-biasedness (Part-1)

Welcome to the series of E-learning modules on Properties of Estimators – Unbiasedness, Relative Efficiency and Consistency.

By the end of this session, you will be able to:

- Explain the properties of estimators
- Explain unbiasedness, asymptotical unbiasedness and associated results
- Explain consistency, Properties and association with unbiasedness
- Explain efficiency and relative efficiency

Many functions of sample observations may be proposed as the estimators of the same parameter.
For example: Either mean or median or mode of the sample values may be used to estimate the parameter teeta of the normal distribution. Naturally, we have to choose one among the various estimators on the basis of certain criteria.

Estimator is accepted or rejected depending on its sampling properties. Evidently best estimator would be the one that falls nearest to the true value of the parameter to be estimated.
In other words, a statistic whose distribution concentrates as closely as possible near the true value of the parameter may be regarded as the best estimator. We expect the estimator to have four desirable properties like consistency, unbiasedness, efficiency and sufficiency to be considered as the best estimator.

First let us see what we mean by Unbiasedness:
The expected value (mean) of the estimate's sampling distribution is equal to the underlying population parameter; that is, there is no upward or downward bias.
Suppose Tn is an estimator of an unknown parameter teeta then, Tn is said to be unbiased for teeta if Expected value of Tn is equal to teeta

Tn is said to be asymptotically unbiased for teeta if Expected value of Tn is equal to teeta as n tends to infinity
If Expected value of Tn is not equal to teeta then Tn is said to be biased for teeta. Bias in estimation is B of teeta is equal to Expected value of Tn minus teeta

Let us see the result one.
The sample mean x bar is an unbiased estimator of the population mean θ
Expected value of x bar is equal to teeta

Now let us see the proof for result one.
Suppose x one, x two, upto, xn is a random sample of size n drawn from a population with mean teeta
Then, the sample mean x bar is equal to summation xi divided by n

Now each of the sample members x one, x two, upto, xn behaves like a random variable because their values in any particular sample depend on chance. For instance, in Simple Random Sampling any of the population members x one, x two, upto, xn may appear at the ith drawing that is, xi is a random variable with the following probability distribution.

Xi takes the values X one, X two upto XN with respective probabilities 1 by N each.
Then we know that Expected value of Xi is equal to X one into 1 by N plus X two into 1 by N plus upto X N into 1 by N plus which is equal to mean of population teeta
Hence, Expected value of sample mean x bar is equal to Expected value of (x one plus x two plus upto xn) by n
Which is equal to Expected value of x one plus Expected value of x two plus upto plus Expected value of xn) by n
Which is equal to (teeta plus teeta plus upto teeta) by n which is equal to teeta
This shows that x bar is an unbiased estimator of teeta

Let us see the result two.
The sample variance s square equal to summation (xi minus x bar) whole square by n is a biased estimator of the population variance sigma square.

Now let us see the proof for result two.
Given s square equal to summation (xi minus x bar) whole square by n
Also we know that summation (xi minus x bar) whole square by sigma square follows Chi-square with (n minus 1) degrees of freedom
summation (xi minus x bar) whole square by sigma square is equal to n into s square by sigma square which follows Chi-square with (n minus 1) degrees of freedom.

Then Expected value of n into s square by sigma square is equal to n minus 1
Which implies Expected value of s square is equal to (n minus 1) into sigma square by n which is not equal to sigma square.
Hence, s square is biased for sigma square.

2. Un-biasedness (Part-2)

Let us see the result three.
An unbiased estimator of the population variance sigma square is given by,
s square is equal to summation (xi minus x bar) whole square by n minus 1

Now let us see the proof for result three.
Let s square is equal to summation (xi minus x bar) whole square by n minus 1
Also we know that ,
Summation (xi minus x bar) whole square by sigma square follows Chi-square with (n minus 1) degrees of freedom
Summation (xi minus x bar) whole square by sigma square which is equal to(n minus 1) into s square by sigma square follows Chi-square with (n minus 1) degrees of freedom.

Expected value of (n minus 1) into s square by sigma square is equal to n minus 1
Which implies Expected value of s square is equal to (n minus 1) into sigma square by (n minus 1) which is equal to sigma square

Hence, s square is an unbiased estimator of the population variance sigma square

Note the distinction between s^2 of result 2 and result 3, in which only the denominators are different. s^2 of result 2 is the variance of the sample observations but s^2 of result 3 is the unbiased estimator of the variance σ^2 in the population.

Let us see the result four.
Show that, generally an unbiased estimator does not possess the invariance property.

Now let us see the proof for result four.

Suppose that Tn is an unbiased estimator of teeta, then the Expected value of Tn is equal to teeta
Let g (dot) be any continuous function of teeta
If Expected value of g of Tn is equal to g of teeta then Tn is invariant with respect to unbiasedness.

Let g of x is equal to x square then g of teeta is equal to teeta square and g of Tn is equal to Tn square
Expected value of g of Tn is equal to Expected value of Tn square which is equal to Variance of Tn plus Expected value of Tn whole square
Which is equal to Variance of Tn plus teeta square which is equal to variance of Tn plus g of teeta
Therefore, Expected value of g of Tn is greater than g of teeta because Variance of Tn is greater than zero
Hence, Expected value of g of tn is not equal to g of teeta . Hence Tn does not possess an invariance property with respect to unbiasedness.

3. Consistency & Properties of Estimators (Part-1)

Let us now see what we mean by Consistency.
A desirable property of a good estimator is that the accuracy should increase when the sample size becomes larger. Larger sample sizes tend to produce more accurate estimates; that is, the sample estimator is expected to come closer to the population parameter as the size of the sample increases.

Let us see the result five.

An estimator is said to be **consistent** if the variance of its sampling distribution decreases with increasing sample size.

This is a good property because it means that if you make the effort to collect data from a larger random sample, you should end up with a more accurate estimate of the population parameter.

Now let us see the proof for result five.

Suppose T_n is an estimator of an unknown parameter teeta. Tn is said to be consistent for teeta if Tn converges to teeta in probability as n tends to infinity
That is Probability of modulus of Tn minus teeta less than epsilon tends to 1 as n tends to infinity
OR Probability of modulus of Tn minus teeta greater than epsilon tends to zero as n tends to infinity
For epsilon greater than zero , yeeta greater than zero
Probability of modulus of Tn minus teeta less than epsilon greater than or equal to 1 minus yeeta as n tends to infinity.

Consistency is a limiting property. Moreover several consistent estimators may exist for the same parameter. For example, in sampling from a Normal population Normal (teeta, sigma square) both the sample mean and the median are the consistent estimators of the population mean teeta.

Let us see the result six.
Necessary and sufficient condition for consistency
Suppose Tn is unbiased for g of teeta and Variance of Tn tends to zero as n tends to infinity then Tn is consistent for g of teeta OR
An estimator Tn is consistent estimator for g of teeta (a function of teeta) if Expected value of Tn is equal to g of teeta and Variance of Tn tends to zero as n tends to infinity.

Now let us see the proof for result six.
Expected value of Tn is equal to g of teeta and Variance of Tn tends to zero as n tends to infinity

Applying Tchebychev’s inequality
Probability of modulus of Tn minus g of teeta less than epsilon greater than or equal to 1 minus variance of Tn by epsilon square
Taking limit as n tends to infinity we get,

Limit as n tends to infinity Probability of modulus of Tn minus g of teeta less than epsilon greater than or equal to 1 minus limit as n ends to infinity variance of Tn by epsilon square which is equal to 1
Which implies Tn tends to g of teeta in probability as n tends to infinity
Therefore, Tn is consistent for g of teeta.

Let us see the result seven.
Consistent Estimators need not be unbiased
Let s square is equal to summation (xi minus x bar) whole square by n
Also we know that,
Summation (xi minus x bar) whole square by sigma square follows Chi-square with (n minus 1) degrees of freedom
Summation (xi minus x bar) whole square by sigma square which is equal to n into s square by sigma square follows Chi-square with (n minus 1) degrees of freedom.

Now let us see the proof for result seven.
Expected value of n into s square by sigma square is equal to n minus 1
And variance of n into s square by sigma square is equal to two into n minus 1
implies Expected value of s square is equal to(n minus 1) into sigma square by (n) which is not equal to sigma square
Hence, s square is biased for sigma square.

But limit as n tends to infinity Expected value of s square is equal to sigma square
Variance of n into s square by sigma square is equal to two into n minus 1
implies variance of s square is equal to two into (n minus 1) into sigma to the power four by n square which tends to zero as n tends to infinity
Hence, s square is equal to summation (xi minus x bar) whole square by n
is a consistent estimator but not an unbiased estimator of the population variance.

4. Consistency & Properties of Consistent Estimators (Part-2)

Let us see the result Eight.
Unbiased estimators need not be consistent.
Now let us see the proof for result eight.
Consider x one, x two, upto x five taken from a Normal population with mean teeta and variance sigma square
That is, xi follows Normal with mean teeta and variance sigma square
Let x bar is equal to summation i runs from 1 to 5, xi by five then x bar follows Normal with mean teeta and variance sigma square by five.

Hence, Expected value of x bar is equal to teeta and variance of x bar is equal to sigma square by 5
That is, a sample mean is unbiased for the population mean teeta

Now the variance of x bar is equal to sigma square by 5 which does not tend to zero as n tends to infinity
Hence, x bar is unbiased is not consistent for teeta.

Let us see the result nine.

Consistent estimators possess invariance property.

Suppose Tn is a consistent estimator of teeta and h of teeta is a continuous function of teeta then h of Tn is consistent for h of teeta.

Now let us see the proof for result eight.
Since, Tn is consistent for teeta then Tn convergers to teeta in probability as n tends to infinity
That is, for epsilon greater than zero , yeeta greater than zero
Probability of modulus of Tn minus teeta less than epsilon greater than or equal to 1 minus yeeta as n tends to infinity
Since, h(dot) is a continuous function , for every epsilon greater than zero , however small there exists a positive number epsilon one such that
Modulus of h of Tn minus h of teeta less than epsilon one whenever modulus of tn minus teeta is less than epsilon.

That is, modulus of tn minus teeta is less than epsilon implies Modulus of h of Tn minus h of teeta less than epsilon one.

For two events A and B
If A implies B then A is the subset or equal to B which implies Probability of A less than or equal to Probability of B which implies P of B greater than or equal to P of A
Probability of modulus of h of Tn minus h of teeta less than epsilon one greater than or equal to Probability of modulus of tn minus teeta is less than epsilon

Probability of modulus of h of Tn minus h of teeta less than epsilon one is greater than or equal to 1 minus yeeta
Which implies h of Tn tends to h of teeta in probability as n tends to infinity
h of Tn is consistent for h of teeta

Let us see the Properties of Consistent Estimators:
There are 5 properties of consistent estimators
1. For any distribution sample mean is the consistent estimator for the population mean
2. An consistent estimator need not be necessarily be unbiased
3. An unbiased estimators need not be consistent
4. A consistent estimator with finite mean is usually asymptotically unbiased
5. Consistent estimators have invariance property

5. Efficiency

Let us see what we mean by Efficiency.

It is possible that there may be several consistent estimators for the same population parameter.
For example, in case of Normal population sample mean and the sample median are both consistent estimators of the population mean. Thus, it is necessary to have a criterion to decide a better estimator within the class of consistent estimators.

While there are many consistent estimates of the same parameter, the most efficient has a sampling distribution with the smallest variance. Good estimate has smaller standard error than other estimates.
In order to make a choice among the consistent estimators, we have to introduce the idea of 'efficiency'.

Of the two consistent estimators for the same parameters, the statistic with the small sample variance is said to be 'more efficient '.
Thus, if t and t dash are both consistent estimators of teeta and Variance of t is less than variance of t dash then t is said to be more efficient than t dash in estimating teeta.

If a consistent estimator exists whose sampling variance is less than that of any other consistent estimator, it is said to be most efficient and it provides a standard for the measurement of efficiency of a statistic.

Further a relative efficiency of T one with respect to T two is defined as
E of T one , T two is equal to Variance of T two by variance of T one

a) If E of T one,T two is equal to 1 then the both T one and T two are equally efficient
b) If E of T one , T two is greater than 1 then T one is more efficient than T two
c) If E of T one , T two is less than 1 then T two is more efficient than T one

Let us see the result ten.

Show that while sampling from a Normal population sample mean is more efficient in estimating population mean than sample median

Now let us see the proof for result ten.

Given xi follows Normal mue , sigma square
X bar follows Normal mue , sigma square by n
Sample median m follows Normal mue , 1 by 4 into (n minus 1) into (f of mue) whole square
E of x bar equals to mue tends to mue as n tends to infinity
Variance of x bar equal to sigma square by n tends to zero as n tends to infinity
Hence x bar tends to mue in probability as n tends to infinity

Variance of m is equal to 1 by 4 into (n minus 1) into (f of mue) whole square
Which is equal to 1 by 4 into (n minus 1) into (1 by sigma into root 2 phi) whole square

Equals to sigma square into 2 phi by 4 into (n minus 1) tends to zero as n tends to infinity
Hence, m tends to mue in probability as n tends to infinity
In sampling form a Normal population both the sample mean and sample median are the consistent estimators of the mean mue
Variance of m is equal to sigma square into phi by 2 into (n minus 1) approaximately equal to sigma square into phi by 2 n which is equal to phi by 2 into variance of x bar
Which implies variance of m is greater than variance of x bar.

Since Variance of x bar is smaller than Variance of m, mean is more efficient than Median in estimating the population parameter mue.
Therefore, sample mean is more efficient than sample median

How well a specific estimator satisfies each of these criteria depends very much on the details of the population distribution.

Thus, we may identify the “best estimator” as the one with:
- Least bias (unbiased)
- consistent
- Minimum variance of estimation error

Here’s a summary of our learning in this session, where we understood:
- The required properties of estimators
- The unbiasedness , asymptotic unbiasedness and related results
- The consistency , associated results and properties
- The efficiency and relative efficiency of estimators