1. Introduction

Welcome to the series of E-learning modules on Practical - Estimation by the Method of Moments. In this module, we are going to cover the basic principle of method of moments and problems on estimating certain population parameters by the method of moments.

By the end of this session, you will be able to:

- Explain the basic principle of method of moments
- Explain the procedure to estimate certain parameters of the population by the method of moments

Basic principles of methods of moments

In statistics, the method of moments is a method of estimation of population parameters such as mean, variance, median, etc. (which need not be moments). The quantities to be estimated can be obtained by equating sample moments with unobservable population moments and solving the equations.

Procedure followed in the method of moments: The moments of the empirical distribution are determined (the sample moments), which are equal to the number of parameters to be estimated. Then, they are equated to the corresponding moments of the probability distribution, which are the functions of unknown parameters. The system of equations thus obtained is solved for the parameters and the solutions are the required estimates.

Scientifically, the method of moments is carried out as follows:
In case of frequency distribution of the sample observations, r^{th} sample moment mr dash is given by mr dash is equal to summation fi into xi to the power r by N. Thus, the method of moments consists in equating r^{th} raw moments about the origin in the population to the r^{th} raw moments about the origin in the sample by giving values r is equal to one, two, etc. and obtaining various equations containing parameters and solving these equations to obtain the estimate of the parameters.
Hence, from the method of moments, unknown parameters are estimated such that mu one dash is equal to m one dash, mu two dash is equal to m two dash etc.

2. Problems on Method of Moments

Problem 1:

Figure 1

No. of heads	0	1	2	3	4	5	6
Frequency	7	64	140	210	132	75	12

A set of 6 similar coins are tossed six hundred and forty times with the following results. Estimate the probability of head by the method of moments.

Solution:
Let X denote the number of heads, n denote the number of tosses and p be the probability of getting a head. Since only one parameter is unknown, we need to find mu one dash only. But for a Binomial variable with parameters n and p
We know that for a Binomial distribution
Mu one dash is equal to n into p implies p is equal to mu 1 dash by n which is equal to mean by n

Now, from the method of moments unknown parameter p is estimated such that mu one dash is equal to m one dash
Hence, a moment estimator of p is p cap is equal to mu 1 cap dash by n, which is equal to m 1 dash by n which is equal to sample mean by n

Sample mean is equal to x bar, which is equal to summation fi xi by N which is equal to one thousand nine hundred and forty nine by six hundred and forty which is equal to three point zero four five
P cap is equal to m1 dash by n, is equal to sample mean by n which is equal to three point zero four five by 6, which is equal to zero point five zero seven six
Hence, probability of getting a head is zero point five zero seven six.

Problem 2:

Figure 2

Central value	30	35	40	45	50	55	60
Frequency	5	9	15	32	28	10	8

Assuming the distribution to be normal, estimate the parameters of the distribution for the following data by the method of moments.

Solution:

Let mu and sigma square be the parameters of the Normal population. Since two parameters are unknown, we need to find mu one dash and mu two dash. We know that, mu 1 dash is equal to mu and sigma square is equal to mu 2 dash minus mu 1 dash square

By the method of moments, the estimation of population moments are obtained by equating

Mu1 dash is equal to m1 dash, which is equal to sample mean

Sigma cap square is equal to m2 dash minus m1 dash square, which is equal to a sample variance

Is equal to 1 by N into summation f into x square minus summation f into x by N whole square

Therefore, the moment estimate of mu is equal to mu cap which is equal to summation f into x by N, which is equal to four thousand five hundred and eighty five by hundred, which is equal to forty five point eight five

sigma cap square is equal to summation f into x square by N minus summation f into x by N whole square, which is equal to two one five eight seven five by hundred minus (forty five point eight five) square, which is equal to fifty six point five two seven five

Therefore, moment estimate of mu is equal to forty five point eight five

Moment estimate of sigma square is equal to fifty six point five two seven five

3. Problems on Estimating the Parameter α and θ

Problem 3:
For the following distribution, estimate the parameter alpha and theta from the data given below:
f of (x, alpha, theta) is equal to x to the power alpha e to the power minus x by theta divided by alpha factorial theta to the power alpha plus 1
Observations: 0 point 8, 1 point 2, 2 point 1, 1 point 3 4, 1 point 3 1, 1 point 7 8, 2 point 0 8, 0 point 0 2

Solution:
We have to estimate alpha and theta based on the given sample observations. Hence, we need to find mu1 dash and mu2 dash
Mu1 dash is equal to integral from 0 to infinity x into f of (x, alpha, theta) dx, which is equal to integral from 0 to infinity x into x to the power alpha into e to the power minus x by theta divided by alpha factorial theta to the power alpha plus 1 dx which is equal to (alpha plus 1) into theta. Call this as (1)

Mu2 dash is equal to integral from 0 to infinity x square into f of (x, alpha, theta) dx, which is equal to integral from 0 to infinity x square into x to the power alpha e to the power minus x by theta divided by alpha factorial theta to the power alpha plus 1 dx, which is equal to (alpha plus 1) into (alpha plus 2) into theta square. Call this as (2)
Equation (2) by equation (1) implies
Mu2 dash by mu1 dash is equal to (alpha plus 2) into theta. Call this as (3)

Equation (3) minus equation (1) implies
Mu2 dash by mu1 dash minus mu1 dash is equal to (alpha plus 2) into theta minus (alpha plus 1) into theta, which is equal to theta, which implies theta cap is equal to Mu2 dash minus Mu1 dash square by mu1 dash
Substituting this value of theta in equation (1)
Mu1 dash is equal to (alpha plus 1) into (Mu 2 dash minus Mu 1 dash square by mu 1 dash, which implies alpha is equal to mu 1 dash square by Mu2 dash minus Mu1 dash square minus 1

Moment estimators of theta and alpha are
Theta cap is equal to (Mu2 dash minus Mu1 dash square by mu1 dash), which is equal to (m2 dash minus m1 dash square by m 1 dash)
Which is equal to summation xi square by n minus (summation xi by n) whole square by summation xi by n, which is equal to 2 point one eight seven one minus one point seven six five seven divided by 1 point three two eight eight, which is equal to zero point three one seven one

Alpha cap is equal to mu1 cap dash by theta cap minus 1, which is equal to m1 dash by theta cap minus 1, which is equal to summation xi by n by theta cap minus 1, which is equal to 1

point three two eight eight by zero point three one seven one minus 1, which is equal to three point one nine zero three
Therefore, moment estimate of theta is equal to zero point three one seven one and
Moment estimate of alpha is equal to three point one nine zero three

Problem 4:
For the following observations, obtain a moment estimator of theta of a uniform population with parameters 0 and theta.
The observations are:
2 point 5, 1 point 6 3, 5 point 2 1, zero point 8 2, 6 point 3 1, 4 point 3 4, 1 point 5 2, 3 point 6 2, 4 point 7 3, 3 point 9 5, 5 point 6 3, 6 point 5

Solution:
We know that when xi follows uniform distribution with range zero and theta
The probability density function f of x is equal to one divided by theta while xi can range from zero to theta
The first population moment mu one dash is equal to theta by two
Now, from the method of moments, unknown parameter theta is estimated such that mu one dash is equal to m one dash

But m one dash is equal to summation xi by n, i runs from one to n, which is equal to x bar
Hence, mu one dash is equal to theta by two, which is equal to m one dash, which is equal to x bar
Therefore, moment estimator of theta is given by theta cap is equal to two into x bar, which is equal to 2 into summation xi by n, which is equal to 2 into forty six point seven six by twelve, which is equal to seven point seven nine three three
Hence, moment estimate of theta is seven point seven nine three three

4. Problems on Moment Estimator (Part-1)

Problem 5:
Find the moment estimator of theta in the Beta population of second kind with parameters 1 and theta. Also, find the estimate of theta based on the observations 1 point 4, 2 point 6, 3 point 1, 0 point 9, 1 point 5, 2 point 4, 2 point 5.

Solution:
Xi follows beta two distribution with parameters 1 and theta
f of xi , theta is equal to 1 by beta (1, theta) x to the power 1 minus 1 by (1 plus x) to the power theta plus 1, which is equal to 1 by beta (1, theta) into 1 by (1 plus x) to the power theta plus 1
When Xi follows beta two distribution with parameters m and n
f of xi, m, n is equal to 1 by beta (m, n) x to the power m minus 1 by (1 plus x) to the power m plus n, 0 less than x less than infinity

Mu r dash is equal to Expected value of x to the power r which is equal to integral of x to the power r into 1 by beta (m,n) into x to the power m minus 1 by (1 plus x) to the power m plus n dx
Which is equal to 1 by beta (m,n) into integral of x to the power m plus r minus 1 by (1 plus x) to the power m plus n dx
Which is equal to 1 by beta (m,n) into integral of x to the power m plus r minus 1 by (1 plus x) to the power m plus r plus n minus r dx
Which is equal to 1 by beta (m,n) into beta (m plus r, n minus r)
Which is equal to Gamma (m plus n) by Gamma m into Gamma n into Gamma (m plus r) Gamma (n minus r) by Gamma (m plus n) which is equal to Gamma (m plus r) Gamma (n minus r) by Gamma (n) into Gamma (m)

Putting r is equal to 1, m is equal to 1, n is equal to theta
We get,
Mu 1 dash is equal to Gamma (1) into gamma (theta minus 1) by Gamma (1) into Gamma (theta), which is equal to 1 by theta minus 1
Mu 1 dash is equal to m 1 dash, which is equal to x bar, which implies 1 by theta minus 1 is equal to x bar, which implies theta is equal to 1 plus x bar by x bar

Hence, the moment estimator of theta is theta cap which is equal to 1 plus x bar by x bar
X bar is equal to summation xi by n, which is equal to fourteen point four by seven, which is equal to 2 point zero five seven
Theta cap is equal to 1 plus x bar by x bar, which is equal to 1 plus 2 point zero five seven by 2 point zero five seven, which is equal to one point four eight six

5. Problems on Moment Estimator (Part-2)

Problem 6:
Given the observations from uniform distribution (theta 1 minus theta 2, theta 1 plus theta 2), obtain the moment estimates of theta 1 and theta 2
1 point 5, minus 1 point 2, 2 point 6, 0 point 6, 0, minus2 point 5, 3 point 8, minus1 point 7, 0 point 8, 0 point 5

Solution:
We know that when xi follows Uniform distribution with range (theta 1 minus theta 2, theta 1 plus theta 2)
The probability density function f of x is equal to one divided by (theta 1 plus theta 2 minus (theta 1 minus theta 2)), which is equal to 1 by 2 into theta 2, while xi can range from (theta 1 minus theta 2 to theta 1 plus theta 2)
We can find the estimates of theta 1 and theta 2 based on the given sample observations. Hence, we need to find mu 1 dash and mu 2 dash
Mu 1 dash is equal to integral from theta 1 minus theta 2 to theta 1 plus theta 2 of x f of (x, theta 1, theta 2) dx, which is equal to integral from theta 1 minus theta 2 to theta 1 plus theta 2 of x into 1 by 2 theta 2 dx, which is equal to theta 1

Mu 2 dash is equal to integral from theta 1 minus theta 2 to theta 1 plus theta 2 of x square into f of (x, theta 1, theta 2) dx, which is equal to integral from theta 1 minus theta 2 to theta 1 plus theta 2 of x square into 1 by 2 theta 2 dx, which is equal to theta 2 square plus 3 theta 1 square by 3
By substituting mu 1 dash is equal to theta 1
We get, theta 2 is equal to square root of 3 into (mu 2 dash minus mu 1 dash square)
By the method of moments, mu 1 dash is equal to m 1 dash is equal to x bar and mu 2 dash is equal to m 2 dash

Therefore, theta 1 cap is equal to x bar is equal to summation xi by n is equal to zero point four four
theta 2 cap is equal to square root of 3 into (m2 dash minus m1 dash square) which is equal to square root of 3 into (summation xi square by n minus x bar square), which is equal to three point one six two eight
Therefore, the moment estimate of theta 1 is zero point four four and
The moment estimate of theta 2 is three point one six two eight.

Problem 7
The lives of 5 tyres is three thousand five hundred and twenty, four thousand and hundred, four thousand four hundred and seventy, three thousand eight hundred and sixty and four thousand one hundred and fifty kilo meters respectively. If life of the tyres has a p.d.f, f of x is equal to theta e to the power minus theta x, x greater than zero, then find the moment estimator of theta.

Solution
We can find the estimate of theta by using the moment Mu 1 dash is equal to integral of x into f of (x, theta) dx, which is equal to integral of x into theta e to the power minus theta x dx which is equal to 1 by theta
Therefore theta cap is equal to 1 by mu 1 dash
By the method of moments, a moment estimate of theta is obtained such that mu 1 dash is equal to m 1 dash is equal to x bar
Therefore, moment estimator of theta is given by theta cap is equal to 1 by m 1 dash, which is equal to 1 by x bar, which is equal to point zero zero zero two five

Here's a summary of our learning in this session, where we have understood:

- The procedure of method of moments
- The practical problems related to discrete and continuous populations