1. Introduction

Welcome to the series of E-learning modules on Practicals on Estimation by the Method of Maximum Likelihood. In this module we are going cover the basic principle of method of maximum likelihood-Problems to demonstrate estimation certain population parameters by the method of maximum likelihood.

By the end of this session, you will be able to:

- Understand the principle of method of maximum likelihood
- Describe the procedure to estimate certain parameters of the population by the method of maximum likelihood

In statistics, maximum-likelihood estimation (MLE) is a method of estimating the parameters of a statistical model which is generally used in estimation.
When applied to a data set and given a statistical model, maximum-likelihood estimation provides estimates for the model's parameters.

In case the sample observations are independent, the likelihood function happens to be the product of the density functions of the random observations. If x one, x two, etc, till xn are n independent and identically distributed observations, from a population with an unknown parameter theta, then the likelihood function of the random observations denoted by
L is equal to L of (x one, x two, etc, xn, theta) is given by
L is equal to L of (x one, x two, etc till xn, theta) is equal to product of f of (xi, theta)
where f of (xi, theta) is the probability density function of the population.

The method of maximum likelihood is the one in which for a given set of values x one, x two, etc, xn, an estimator of theta is found that maximizes L. Thus if there exists theta cap, a function of x one, x two, etc till xn for which L is maximum for variations in theta. Then theta cap is the maximum likelihood expectation of theta if
Delta L by delta theta is equal to zero and Delta square L by delta theta square is less than zero.

In practice it is often more convenient to work with the logarithm of the likelihood function, called the log-likelihood.
Since log L attains maximum when L attains maximum the solution of the equation,
Log L of (x one, x two, etc till xn, theta) is equal to summation log f of (xi, theta)
Delta log L by delta theta is equal to zero and Delta square log L by delta theta square is less than zero also gives the maximum likelihood expectation of theta.

2. Illustrative Examples

Illustrative Examples:
Problem 1:
A random variable X takes values zero and one. Obtain on the basis of a random sample of size n, the maximum likelihood expectation for the probability of zero. If the following were the observations, what is the estimate of the probability of zero?
The Observations are: zero, zero, one, zero, one, zero, one, one, one, zero, one, zero, zero, one, one, zero, one, zero, zero, and zero.

Solution:
Let x one, x two, etc, xn be a random sample of size n drawn from a Bernoulli population. Let p denote the probability of zero. Then (one minus p) is the probability of one. Out of n observations let there be m number of zeroes. Thus there will be n m number of one's.

Therefore the likelihood function is
L is equal to L of (x one, x two, etc, xn, p) which is equal to product of f of (x i , p) ,i runs from one to n which is equal to p to the power m into (one minus p) to the power (n minus m)
LN L is equal to m into LN p plus (n minus m) into LN(one minus p)

Differentiating with respect to 'p' and equating it to zero
Delta LN L by delta p is equal to zero implies m by p minus (n minus m) by (one minus p) is equal to zero implies m into (one minus p) minus n into p plus m into p is equal to zero which implies p cap is equal to m by n.

Therefore estimate of p is number of zeroes divided by total number of observations.
Here n is equal to twenty and m is equal to eleven
Therefore p cap is equal to m by n which is equal to eleven by twenty which is equal to zero point five, five.
Hence maximum likelihood estimate of the probability of zero is zero point five, five.

Problem 2:
Obtain maximum likelihood expectation of theta based on the sample observations for the following function f of (x , theta) is equal to one by theta into e to the power minus x i by theta , x i is greater than zero.
The observations are as follows:
ten, three point seven two, twelve point one, zero point seven eight, five point three one, two point nine eight, eight point eight, eight , nine point three.

Solution:
Let x one, x two, etc, xn be a random sample of size n drawn from the given population with probability density function. Given f of (x , theta) is equal to one by theta into e to the power minus x i by theta
L is equal to L of (x one, x two, etc, xn, theta) which is equal to product of f of (x i , theta) ,i runs from one to n which is equal to one by theta to the power n into e to the power minus

summation x i by theta
LN L is equal to minus n into LN theta minus summation x i by theta.

Differentiating with respect to 'theta' and equating it to zero
Delta LN L by delta theta is equal to zero implies minus n by theta plus summation x i by theta square is equal to zero implies minus n plus summation x i by theta equal to zero implies theta is equal to summation x i by n which is equal to x bar.
Delta square LN L by delta theta square is equal to n by theta square minus two into summation x i by theta cube which is equal to minus n by theta square which is less than zero.
Therefore Maximum likelihood estimation of theta is theta cap which is equal to summation x i by n which is equal to fifty three point zero seven by eight equal to six point six three, three seven five.

3. Examples (Part – 1)

Problem 3:
Find the M. L. E of theta in the following Uniform distribution U(0, theta)
F of (x, theta) is equal to one by theta , zero less than x less than theta using the sample observations
two point five, one point six three, five point two one, zero point eight two, six point three one, four point three four, one point five two, three point six two, four point seven three, three point nine five, five point six three, six point five.

Solution:
Let x one, x two, etc, xn be a random sample of size n drawn from an Uniform population with probability density function
F of (x, theta) is equal to one by theta, x varies between zero and theta.
L is equal to L of (x one, x two, etc, xn, theta) is equal to product of f of (xi, theta). This is equal to (one by theta) to the power n.
Log L is equal to minus n into log theta.

Delta log L by delta theta equal to zero implies minus n by theta is equal to zero which implies theta cap is equal to infinity or n is equal to zero Which is meaningless.
Hence we use the basic principle of the maximum likelihood estimation
maximum likelihood expectation is that value of theta for which L is maximum
L is maximum when one by theta to the power n is maximum. That is when theta to the power n is minimum.
That is when theta is minimum.

But x varies between zero and theta. This implies,
Zero less than or equal to x (one) less than or equal to x (two) less than or equal to etc less than or equal to x (n) less than or equal to theta.
Minimum value of theta is x n, the maximum of the observations.
Therefore maximum likelihood expectation of theta is theta cap is equal to x n, which is equal to six point five, the maximum of the observations in the given set of data.

Problem 4:
The following observations were obtained from Uniform (alpha, beta) distribution. Find the maximum likelihood estimates of alpha and beta. The observations are:
minus two, zero point one, minus one point two, four point six, minus two point five, zero, zero point five, two point eight, minus one point six, one point five.

Solution:
The given random sample is drawn from Uniform population with parameters alpha and beta.
Hence when xi follows Uniform (alpha , beta), the probability density function is given by
F of x is equal to one by beta minus alpha, alpha less than or equal to x less than or equal to

beta.

L is equal to L of (x one, x two, etc, xn, alpha, beta) is equal to product of f of (xi, alpha, beta) Equal to (one by beta minus alpha) to the power n.
Log L is equal to minus n into log (beta minus alpha).
Delta log L by delta beta equal to zero implies minus n by beta minus alpha is equal to zero which implies beta minus alpha is equal to infinity Which is meaningless.

Hence the usual method of finding maximum likelihood expectation fails. Now we use the basic principle of the maximum likelihood estimation.
Maximum likelihood expectations of alpha and beta are that values for which L is maximum.

L is maximum when one by (beta minus alpha) to the power n is maximum. That is when (beta minus alpha) to the power n is minimum.
That is when (beta minus alpha) is minimum.
(beta minus alpha) is minimum when alpha is maximum and beta is minimum.

But alpha less than or equal to x less than or equal to beta implies alpha less than or equal to x(one) less than or equal to x(two) less than or equal to etc till less than or equal to x(n) less than or equal to beta.
Minimum value of beta is xn, the maximum of the observations and the maximum value of alpha is x one, the minimum of the observations.

Therefore maximum likelihood expectation of alpha is x one beta is x n. Hence in the given data minimum of the observations is minus two point five and maximum of the observations is four point six.
Therefore maximum likelihood estimate of alpha is x one which is minimum of the observations which is equal to minus two point five and maximum likelihood estimate of beta is x n, which is maximum of the observations which is equal to four point six.

4. Examples (Part - 2)

Problem 5:

The following sample is taken from a Normal population N (theta, sigma square). Find the maximum likelihood estimators for theta and sigma square and hence their respective estimates. The observations are :fifty two point seven, fifty nine, sixty two, sixty two point seven, forty seven, fifty two, fifty five, fifty six, sixty five, and forty two.

Solution:

Let x one, x two, etc till x n be a random sample of size n drawn from a given population.
L is equal to L of (x one, x two, etc, xn, theta) is equal to product of f of (xi, theta)
Which is equal to product of one by sigma into root two pi into exponential to the power (minus one by two sigma square into (xi minus theta), the whole square.
Which is equal to one by sigma into root two pi) to the power n into exponential to the power (minus one by two sigma square summation (xi minus theta) whole square.

LN L is equal to minus n by two into LN sigma square minus n into LN root two pi minus one by two sigma square into summation (xi minus theta) whole square. Maximum likelihood estimates of theta and sigma square when both are unknown are: delta LN L by delta theta is equal to zero implies summation (xi minus theta) by sigma square is equal to zero implies theta cap is equal to summation xi by n which is x bar. Call this as equation one.

Delta LN L by delta sigma square is equal to zero implies minus n plus summation (xi minus theta) whole square by sigma square is equal to zero.
Implies minus n plus summation (xi minus theta cap) whole square by sigma square is equal to minus n plus summation (xi minus x bar) whole square by sigma square is equal to zero.
From equation one, Which implies sigma cap square is equal to summation (xi minus x bar) whole square by n.

Therefore maximum likelihood expectation of theta is, theta cap is equal to summation x i by n, which is equal to x bar.
The maximum likelihood expectation of sigma square when theta is unknown is sigma cap square is equal to summation (xi minus x bar) whole square by n .

In the given problem the maximum likelihood estimate of theta is:
For the given problem the maximum likelihood estimate of theta is, theta cap is equal to summation x i by n. Which is equal to five hundred fifty three point four by ten which is equal to fifty five point three four and the maximum likelihood estimate of sigma square when theta is unknown is, sigma cap square is equal to summation (xi minus x bar) whole square by n, equal to four hundred and seventy one point four two four by ten which is equal to forty seven point one four two four.

5. Examples (Part - 3)

Problem 6:
From a population with density function, f of x is equal to one by two into 'e' to the power minus modulus of (xi minus theta), the following ten random observations are drawn.
Ten point seven, two point one three, three point four seven, twelve, five point four, zero point zero seven, minus two, seven point five , minus zero point one, four point four five.
Obtain Maximum likelihood estimate of theta.

Solution:
Let *x* one, x two, etc till xn be a random sample of size n drawn from a given population.
L is equal to L of (x one, x two, etc till xn, theta) is equal to product of f of (xi, theta) , i runs from one to n which is equal to product of one by two into 'e' to the power minus modulus of (xi minus theta)
Which is equal to (one by two) to the power n into 'e' to the power minus summation modulus of (xi minus theta).

LN L is equal to minus n into LN into two minus summation modulus of (x i minus theta).
Since the function minus summation modulus of (x i minus theta)
is not differentiable, we use the definition to find the maximum likelihood expectation of theta . That is, the value of theta for which L is maximum. L is maximum when 'e' to the power minus modulus of (xi minus theta)
is maximum .
L is maximum when summation modulus of (x i minus theta) is minimum.

L is maximum when one by n into summation modulus of (x i minus theta) is minimum.
L is maximum when Mean deviation about theta is minimum. But mean deviation about theta is minimum when theta is equal to Median. Therefore the maximum likelihood expectation of theta is theta cap is equal to Median which can be obtained as follows :

Arranging the given observations in an ascending order we get
Minus two, minus zero point one, zero point zero seven, two point one three, three point four seven, four point four five, five point four, seven point five, ten point seven and twelve.
Median is the value of (n plus one) by two)th item is equal to Value of five point fifth item.

Which is equal to value of fifth item plus (zero point five) into (Value of sixth item minus Value of fifth item).
Which is equal to three point four seven plus (zero point five) into (four point four five minus three point four seven) which is equal to three point nine six.
Therefore the maximum likelihood expectation of theta is theta cap which is
Equal to median, this is equal to three point nine six.

Problem 7:

From a population with density
F of (x, alpha, beta) is equal to alpha x to the power (alpha minus one) by (one plus x) to the power alpha plus one, where x varies between zero and infinity and alpha is greater than zero.
The following ten random observations are drawn. Obtain Maximum likelihood estimate of alpha.
The observations are seventy five, sixty eight, fifty five, sixty two, fifty, eighty two, seventy seven, sixty, seventy two and fifty.

Solution:
L is equal to product of f of (xi , alpha) which is equal to alpha to the power n product of xi to the power alpha minus one by product of (one plus xi) to the power (alpha plus one)
LN L equal to n LN into alpha plus (alpha minus one) into summation LN xi minus (alpha plus one) into summation LN(one plus x i).

Differentiating with respect to alpha and then equating to zero we get, alpha cap is equal to n divided by summation LN into (one plus xi) minus summation LN xi.
Therefore the maximum likelihood expectation of alpha is,
Alpha cap is equal to n by summation LN into (one plus xi) minus summation LN into xi. This is equal to ten divided by forty one point seven, seven three nine minus forty one point six one seven zero, which is equal to sixty three point seven three four nine.

Here’s a summary of our learning in this session where we have :

- Discussed the principle of method of maximum likelihood estimation
- Explained illustrative examples to obtain maximum likelihood expectation of the unknown parameters of the population