Frequently Asked Questions

1. Explain briefly how do we find the M.L.E of the parameters of the population under consideration?

Answer:
The method of maximum likelihood is the one in which for a given set of values (x_1, x_2… x_n) an estimator of θ is found that maximizes L. Thus if there exists $\widehat{\theta}$, a function of (x_1, x_2, …, x_n) for which L is maximum for variations in θ . Then $\widehat{\theta}$ is the m.l.e of θ then

$$\frac{\partial L}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 L}{\partial \theta^2} < 0$$

In practice it is often more convenient to work with the logarithm of the likelihood function, called the log-likelihood:

Since log L attains maximum when L attains maximum the solution of the equation

Log L (x_1, x_2, …, x_n ,θ)= $\sum_{i=1}^{n} \log f(xi, \theta)$

$$\frac{\partial \log L}{\partial \theta} = 0 \text{ with } \quad \frac{\partial^2 \log L}{\partial \theta^2} < 0$$ also gives the m.l.e. Of θ

2. Find the m.l.e of θ in the following Uniform distribution U(0,θ)

F(x,θ) = 1/θ, 0<x<θ using sample observations 0.5, 2.63, 8.21, .82, 10.31, 5.34, 2.52, 4.62, 5.73, 4.95, 6.63,7.5

Solution:

Let x_1, x_2, …, x_n be a random sample of size n drawn from an Uniform population with probability density function

F(x,θ) = 1/θ, 0<x<θ

L = L(x_1, x_2, …, x_n ,θ)= $\prod_{i=1}^{n} f(x_{i,}\theta)$ = $(1/\theta)^n$

Log L = -n log θ

$$\frac{\partial \log L}{\partial \theta} = 0 \Rightarrow -\frac{n}{\theta} = 0 \Rightarrow \hat{\theta} = \infty \quad \text{or } n=0$$

This is meaningless.

Hence we use the basic principle of the maximum likelihood estimation

M.L.E is that value of θ for which L is maximum

L is maximum when $1/\theta^n$ is maximum

That is when θ^n is minimum

That is when θ is minimum

But $0 \le x \le \theta \Rightarrow 0 \le x_{(1)} \le x_{(2)} \le ...... \le x_{(n)} \le \theta$

Minimum value of θ is $x_{(n)}$, the maximum of the observations

Therefore m.l.e of θ is $\hat{\theta} = x_{(n)} = 10.31$, the maximum of the observations in the given set of observations

3. Find the M.L.E of θ for the function f(x) = $e^{-(x-\theta)}, x \ge \theta$ using the observations
2.1, 8.3, 4.7, 0.82, 7.13, 5.2

Answer:

Let $x_1, x_2, ..., x_n$ be a random sample of size n drawn from a given population

$$L = L(x_1, x_2, ..., x_n, \theta) = \prod_{i=1}^{n} f(x_i, \theta) = e^{-\sum_{i=1}^{n} (x_i - \theta)}$$

Ln L = $-\Sigma(x_i-\theta)$

$$\frac{\partial \ln L}{\partial \theta} = 0 \Rightarrow n = 0 \quad \text{This is meaningless}$$

Hence we use the basic principle of the maximum likelihood estimation

M.L.E is that value of θ for which L is maximum

L is maximum when $\Sigma(xi-\theta)$ is minimum

That is when θ is maximum

But θ ≤ x $\Rightarrow \theta \leq x_{(1)} \leq x_{(2)} \leq ......$

Maximum value of θ is $x_{(1)}$, the minimum of the observations

Therefore m.l.e of θ is $\hat{\theta} = x_{(1)} = 0.82$, the maximum of the observations in the given set of observations

4. A random variable X takes values 0 and 1. Obtain on the basis of a random sample of size n, the M.L.E for the probability of 0. If the following were the observations what is the estimate of the probability of 0

Observations: 0,1,1,0,1,0,1,1,1,0,1,0,0,1,1,0,1,0,0,0,0,1,1,1,0

Solution:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from a Bernoulli population. Let p denote the probability of 0. Then (1-p) is the probability of 1. Out of n observations let there be m number of zeroes . Thus there will be n-m number of one's.

Therefore the likelihood function is

L = L($x_1, x_2, \ldots, x_n$,p)= $\prod_{i=1}^{n} f(x_{i,} p) = p^m (1-p)^{n-m}$

Ln L = m ln p+(n-m)ln(1-p)

Differentiating with respect to 'p' and equating it to zero

$$\frac{\partial \ln L}{\partial p} = 0 \Rightarrow \frac{m}{p} - \frac{(n-m)}{1-p} = 0 \Rightarrow m(1-p) - np + mp = 0 \Rightarrow \hat{p} = \frac{m}{n}$$

Therefore estimate of p is number of zeroes / total number of observations

Here n = 25 and m=12

Therefore $\hat{p} = \frac{m}{n} = \frac{12}{25} = 0.48$

Hence a maximum likelihood estimate of the probability of 0 is 0.48

5. Obtain M.L.E of θ based on the sample observations for the following function $f(x,\theta) = \theta e^{-\theta xi}$, xi >0.

10, 3.72, 12.1, 0.78, 5.31, 2.98, 8.88, 9.3

Solution:

Given $f(x,\theta) = (\theta) e^{-\theta xi}$

$L = L(x_1, x_2, \ldots, x_n, \theta) = \prod_{i=1}^{n} f(x_i, \theta)$

$$= (\theta)^n e^{-\theta \sum xi}$$

Ln L= n ln θ - θ∑xi

Differentiating with respect to 'θ' and equating it to zero

$$\frac{\partial \ln L}{\partial \theta} = 0 \Rightarrow \frac{n}{\theta} - \sum xi = 0 \Rightarrow \theta = \frac{n}{\sum xi} = \frac{1}{\bar{x}}$$

Therefore m.l.e of θ is $\hat{\theta} = \frac{n}{\sum xi} = \frac{1}{\bar{x}} = \frac{1}{6.63375} = 0.15074$

6. From a population with density

$$f(x, \alpha, \beta) = \frac{\alpha x^{\alpha-1}}{(1+x)^{\alpha+1}} \qquad 0< x< \infty, \alpha >0$$

The following 10 random observations are drawn. Obtain Maximum likelihood estimate of α .

75, 68, 55, 62, 50, 82, 77, 60, 72, 50

Solution:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from a given population

L = L($x_1, x_2, \ldots, x_n$,α)= $\prod_{i=1}^{n} f(x_{i,}\alpha) = \prod_{i=1}^{n} \frac{\alpha x^{\alpha-1}}{(1+x)^{\alpha+1}}$

$$= \frac{\alpha^n \prod_{i=1}^{n} xi^{\alpha-1}}{\prod_{i=1}^{n} (1+xi)^{\alpha+1}}$$

Ln L= n ln α +(α-1)Σln xi-(α+1) Σln(1+xi)

Differentiating with respect to α and then equating to zero

$$\frac{\partial \ln L}{\partial \alpha} = 0 \Rightarrow \frac{n}{\alpha} + \sum \ln xi - \sum \ln(1+xi) = 0 \Rightarrow \hat{\alpha} = \frac{n}{\sum \ln(1+xi) - \sum \ln xi}$$

Therefore the M.L.E of α :

$$\hat{\alpha} = \frac{n}{\sum \ln(1+xi) - \sum \ln xi} = \frac{10}{41.7739 - 41.6170} = 63.7349$$

7. Find the maximum likelihood estimate of θ of the geometric distribution with parameter θ using the data 1,7, 6, 2, 3,1, 2,7,6, 5,7, 1, 9, 8, 1, 2, 3

Answer:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from the Geometric population with probability mass function

F(x,θ) = θ (1- θ)^x

The likelihood function

L = L($x_1, x_2, \ldots, x_n$,θ)= $\prod_{i=1}^{n} f(x_{i,}\theta)$ = θ^n (1-θ)$^{\Sigma xi}$

Log L= n log θ+ Σxi log (1- θ)

$$\frac{\partial \log L}{\partial \theta}=0 \Rightarrow \frac{n}{\theta}-\frac{\sum xi}{(1-\theta)}=0 \Rightarrow n(1-\theta)-\theta\sum xi=0 \Rightarrow \theta=\frac{1}{1+\overline{x}}$$

Hence m.l.e of θ of Geometric population is $\widehat{\theta}=\frac{1}{1+\overline{x}}=0.193$

8. The following sample is taken from a normal population with mean μ and variance σ^{2}. Find the maximum likelihood estimators of θ and σ^2

2.3, 5.4, 0.6, -2.9, -1.6, 1.3, 6.9, -0.7

Answer:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from a given

population

L = L($x_1, x_2, \ldots, x_n$,θ)= $\prod_{i=1}^{n} f(x_{i,}\theta)$ $=\prod_{i=1}^{n}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}(xi-\theta)^2}$

$$=\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(xi-\theta)^2}$$

Ln L = $\frac{-n}{2}\ln\sigma^2-n\ln\sqrt{2\pi}-\frac{1}{2\sigma^2}\sum(xi-\theta)^2$

Maximum likelihood estimate of θ and σ^2 when both are unknown

$$\frac{\partial \ln L}{\partial \theta}=0 \Rightarrow \frac{\sum(xi-\theta)}{\sigma^2}=0 \Rightarrow \widehat{\theta}=\frac{\sum xi}{n}=\overline{x} \text{-------(1)}$$

$$\frac{\partial \ln L}{\partial \sigma^2}=0 \Rightarrow \frac{-n}{} + \frac{\sum(xi-\theta)^2}{\sigma^2}=0 \Rightarrow \frac{-n}{}+\frac{\sum(xi-\widehat{\theta})^2}{\sigma^2}=\frac{-n}{}+\frac{\sum(xi-\overline{x})^2}{\sigma^2}\,0$$

From equation (1)

$$\Rightarrow \hat{\sigma}^2 = \frac{\sum (xi - \bar{x})^2}{n}$$

Therefore m.l.e of θ is $\hat{\theta} = \frac{\sum xi}{n} = \bar{x}$ and the m.l.e of σ^2 when θ is unknown is $\hat{\sigma}^2 = \frac{\sum (xi - \bar{x})^2}{n}$

For the given problem the maximum likelihood estimate of θ is $\hat{\theta} = \frac{\sum xi}{n} = \frac{11.3}{8} = 1.4125$ and the maximum likelihood estimate of σ^2 when θ is unknown is $\hat{\sigma}^2 = \frac{\sum (xi - \bar{x})^2}{n} = \frac{79.60875}{8} = 9.951094$

9. From a population with density

f(x) = $\frac{1}{2} e^{-|xi - \theta|}$

The following 10 random observations are drawn.

10.7, 2.13, 3.47, 12, 5.4, 0.07, -2, 7.5, -0.1, 4.45 Obtain Maximum likelihood estimate of θ

Solution:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from a given population

L = L($x_1, x_2, \ldots, x_n$,θ)= $\prod_{i=1}^{n} f(x_{i,} \theta)$

$$= \prod_{i=1}^{n} \frac{1}{2} e^{-|xi-\theta|}$$

$$= \left(\frac{1}{2}\right)^n e^{-\sum |xi-\theta|}$$

lnL=-n ln2 -$-\sum|xi-\theta|$

Since the function $-\sum|xi-\theta|$ is not differentiable we use the definition to

find the M.L.E of θ. That is the value of θ for which L is maximum

L is maximum when $e^{-\sum |xi-\theta|}$ is maximum

L is maximum when $\sum|xi-\theta|$ is minimum

L is maximum when $\frac{1}{n}\sum|xi-\theta|$ is minimum

L is maximum when Mean deviation about θ is minimum. But mean deviation about θ is minimum when θ=Median. Therefore the M.L.E of θ is $\widehat{\theta}$ = Median which can be obtained as follows

Arranging the given observations in an ascending order we get

-2, -0.1, 0.07, 2.13, 3.47, 4.45, 5.4, 7.5, 10.7, 12

Median is the value of ((n+1)/2)th item= Value of (5.5)th item

= Value of 5^{th} item + (0.5) (Value of 6^{th} item -Value of 5^{th} item)

= 3.47 + (0.5) (4.45 - 3.47)=3.96

Therefore the M.L.E of θ is $\widehat{\theta}$ = Median=3.96

10. The following are the observations from N(-3, σ^2). Find the m.l.e of σ

Observations are:
-4.1, -2.8, 0, 1.3, -5.2, 0.8, -2.6, -0.6, -0.4, 1, -4.8, - 3.6, -2.8, -5.6, -7,

-6.2

Answer:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from a given population

L = L($x_1, x_2, \ldots, x_n$,θ)= $\prod_{i=1}^{n} f(x_{i,}\theta) \quad = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(xi-\theta)^2}$

Here θ is known as - 3 . Hence a maximum likelihood estimator of σ^2 when θ is known is $\hat{\sigma}^2 = \frac{\sum (xi-\theta)^2}{n} = \frac{\sum (xi+3)^2}{16} = 5.0588$

A maximum likelihood estimator of σ when θ is known is √5.0588=2.2492

11. The following observations were obtained from U (α, β) distribution. Find the maximum likelihood estimates of α and β

-2, 0.1, -1.2, 4.6, -2.5, 0, 0.5, 2.8, -1.6, 1.5

Solution:

The given random sample is drawn from Uniform population with parameters α and β.

Hence when xi ~ U(α , β), the probability density function is given by

f(x) = 1/(β-α), α ≤ x ≤β

L = L($x_1, x_2, \ldots, x_n$,α,β)= $\prod_{i=1}^{n} f(x_{i,}\alpha.\beta)$ = $(1/\beta-\alpha)^n$

Log L = -n log (β-α)

$$\frac{\partial \log L}{\partial \beta} = 0 \Rightarrow -\frac{n}{\beta-\alpha} = 0 \Rightarrow \beta-\alpha=\infty$$ This is meaningless

Hence the usual method of finding M.L.E fails. Now we use the basic principle of the maximum likelihood estimation

M.L.E's of α and β is that values for which L is maximum

L is maximum when $1/(\beta-\alpha)^n$ is maximum

That is when $(\beta-\alpha)^n$ is minimum

That is when (β-α) is minimum

(β-α) is minimum when α is maximum and β is minimum

But α ≤ x ≤ β $\Rightarrow \alpha \le x_{(1)} \le x_{(2)} \le ...... \le x_{(n)} \le \beta$

Minimum value of β is $x_{(n)}$, the maximum of the observations and the maximum value of α is $x_{(1)}$, the minimum of the observations

Therefore M.L.E of α is $x_{(1)}$ and β is $x_{(n)}$,

Hence in the given data Minimum of the observations is -2.5 and Maximum of the observations is 4.6. Therefore maximum likelihood estimate of α is $x_{(1)}$ which is minimum of the observations = -2.5 and maximum likelihood estimate of β is $x_{(n)}$, which is maximum of the observations = 4.6

12. Following is a random sample from

f(x, α , β) = β $e^{-\beta(x-\alpha)}$ x>=α

221, 280, 290, 252, 263, 211, 270, 262, 231, 280, 242, 271, 232

Obtain Maximum likelihood estimates of α and β

Answer:

The given random sample is drawn from a population with probability density function

$$f(x,\alpha,\beta) = \beta e^{-\beta(x-\alpha)}, x \ge \alpha, \beta > 0$$

L = L($x_1, x_2, \ldots, x_n$,α,β)= $\prod_{i=1}^{n} f(x_{i,}\alpha.\beta) = (\beta)^n e^{-\sum \beta^{(x-\alpha)}}$

Ln L = n ln β - $\sum \beta^{(x-\alpha)}$

$$\frac{\partial \ln L}{\partial \beta} = 0 \Rightarrow -\frac{n}{\beta} - (\sum xi - n\alpha) = 0 \Rightarrow \widehat{\beta} = \frac{1}{\overline{x} - \alpha}$$

$$\frac{\partial \ln L}{\partial \alpha} = 0 \Rightarrow n\beta = 0 \Rightarrow \beta = 0/n$$ This is meaningless

Hence the usual method of finding M.L.E fails. Now we use the basic principle of the maximum likelihood estimation

M.L.E's of α is that values for which L is maximum

L is maximum when $e^{-\sum \beta^{(x-\alpha)}}$ is maximum

That is when $\sum \beta^{(x-\alpha)}$ is minimum

That is when α is maxiimum

But α ≤ x $\Rightarrow \alpha \le x_{(1)} \le x_{(2)} \le ...... \le x_{(n)}$

The maximum value of α is $x_{(1)}$, the minimum of the observations

Therefore M.L.E of α is $x_{(1)}$ and β is, $\widehat{\beta} = \frac{1}{\overline{x} - \widehat{\alpha}} = \frac{1}{\overline{x} - x_{(1)}}$

Therefore $\widehat{\alpha} = x_{(1)} = 211$ and $\widehat{\beta} = \frac{1}{\overline{x} - \widehat{\alpha}} = \frac{1}{\overline{x} - x_{(1)}} = \frac{1}{245 - 211} = 0.2941$

Therefore MLE of α is 211 and β is 0.2941

13. Find the MLE of α in the distribution $f(x) = \frac{\alpha^4 e^{-\alpha xi} xi^3}{\rceil 4}$, x ≥ 0

Sample observations are: 1.4, 2.6, 3.1, 0.9, 1.5, 2.4, 2.5, 1.9, 2.1, 1.6

Solution:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from a given population

L = L($x_1, x_2, \ldots, x_n$,ɑ)= $\prod_{i=1}^{n} f(x_{i,}\alpha) = \prod_{i=1}^{n} \frac{\alpha^4 e^{-\alpha xi} xi^3}{\rceil 4}$

$$= \frac{\alpha^{4n} e^{-\alpha \sum_{i=1}^{n} xi} \prod_{i=1}^{n} xi^3}{(\rceil 4)^n}$$

Ln L=4 n ln ɑ - ɑ∑xi+3∑ln xi-n ln $\rceil 4$

Differentiating with respect to ɑ and then equating to zero

$$\frac{\partial \ln L}{\partial \alpha} = 0 \Rightarrow \frac{4n}{\alpha} - \sum xi = 0 \Rightarrow \hat{\alpha} = \frac{4n}{\sum xi}$$

Therefore the M.L.E of ɑ :

$$\hat{\alpha} = \frac{4n}{\sum xi} = \frac{4*10}{20} = 2$$

14. Find the M.L.E. of θ from the distribution

$$F(x, \theta) = \frac{\theta}{(1+x)^{1+\theta}} \qquad 0 < x < \infty \quad , \theta > 0$$

From which the following random observations are drawn 12, 16, 14, 13, 18, 12, 15, 14, and 13

Answer:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from a given population

L = L($x_1, x_2, \ldots, x_n$,θ)= $\prod_{i=1}^{n} f(x_{i,}\theta) = \prod_{i=1}^{n} \frac{\theta}{(1+xi)^{1+\theta}} = \frac{\theta^n}{\prod_{i=1}^{n}(1+x_i)^{1+\theta}}$

Ln L = n ln θ- (1+θ)∑ln(1+ xi)

Differentiating with respect to θ and then equating to zero

$$\frac{\partial \ln L}{\partial \theta} = 0 \Rightarrow \frac{n}{\theta} - \sum \ln(1+xi) = 0 \Rightarrow \hat{\theta} = \frac{n}{\sum \ln(1+xi)}$$

Therefore the M.L.E of θ :

$$\hat{\theta} = \frac{n}{\sum \ln(1+xi)} = \frac{10}{27.1469} = 0.3684$$

15. A frequency function is given by f(x, θ) = (1+θ)X^{θ} , θ > 0 , 0 ≤ X ≤1

Random samples of 6 values from the distribution are 0.25, 0.72, 0.37, 0.95, 0.97, 0.84, 0.04, 0.33, 0.99, and 0.08; from these values compute M.L.E. of θ.

Answer:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from a given population

L = L($x_1, x_2, \ldots, x_n$,θ)= $\prod_{i=1}^{n} f(x_{i,}\theta) = \prod_{i=1}^{n} (1+\theta) xi^{\theta} = (1+\theta)^n \prod_{i=1}^{n} x_i^{\theta}$

Ln L = n ln (1+θ)+θ∑ln xi

Differentiating with respect to θ and then equating to zero

$$\frac{\partial \ln L}{\partial \theta} = 0 \Rightarrow \frac{n}{1+\theta} + \sum \ln xi = 0 \Rightarrow \hat{\theta} = -1 - \frac{n}{\sum \ln xi}$$

Therefore the M.L.E of θ :

$$\hat{\theta} = -1 - \frac{n}{\sum \ln xi} = -1 + \frac{10}{9.8285} = 0.0174$$