1. Introduction

Welcome to the series of E-learning modules on Practicals on applications of convergence in probability distribution, Central Limit Theorem and Weak Law of Large Numbers. In this module we are going cover the concepts of Central Limit Theorem, Weak Law of Large Numbers and convergence criteria. We shall also demonstrate the application of these theorems to the practical problems.

By the end of this session, you will be able to know:

- Describe Central Limit Theorem, Weak Law of Large Numbers and convergence criteria
- Understand related problems and the procedure to apply these theorems to practical problems

The sampling distribution of the sample mean, x bar, is approximated by a normal distribution when the sample is, a simple random sample and the sample size, n, is large.

The Central Limit Theorem states that under rather general conditions sums and means of random samples of measurements drawn from a population tend to have an approximately Normal distribution.

Statement of Central Limit Theorem
If X one ,X two, etc till X n, are 'n' independent random variables having the same probability density function, each with expected value of Xi is equal to mu and Variance of Xi is equal to sigma square , for I is equal to one, two , etc till n then, S n is equal to X one plus X two plus etc till plus X n is approximately Normal, with mean 'n' mu and variance 'n' sigma square.
Also,
Z is equal to x bar minus n mu by sigma into root n is asymptotically Normal with mean zero and variance one.

Weak Law of Large Numbers
If a sequence x i is an infinite sequence of independent identically distributed random variables with common mean **mu,** we define 'y' 'n' as the random variable equal to the mean of the first n Xi observations. Then, for any **epsilon,** the probability for a realization of Yn to fall more than **epsilon** away from **mu tends** to zero as 'n' grows without limit.

or No matter how small epsilon , all you have to do, to make the probability for the mean of the first 'n' terms to differ from the mean mu by more than epsilon to be as small as you wish, is to make n large enough.

In the vocabulary of Estimation, the Weak Law of Large Numbers states that, the sample mean is a consistent estimator of the population mean. That is a sample mean converges in probability to the population mean
A sequence 'x' 'n' of random variables converges in probability towards X, if for all epsilon greater than zero. Limit as n tends to infinity, Probability of modulus of (X n minus X) greater than or equal to epsilon is equal to zero.

For the existence of the law we assume the following conditions

i) Expected value of (X i) exists for all i

ii) Bn is equal to Variance of (X one plus X two plus etc till plus X n) exists and

iii) B n by n square tends to zero as n tends to infinity

2. Illustrative Examples - Part 1

Illustrative examples
Problem 1:
An insurance company has ten thousand automobile policy holders. If the expected yearly claim per policy holder is dollar two hundred and sixty with a standard deviation of eight hundred dollars, approximate the probability that the total yearly claim exceeds two point eight million dollars.

Solution:
Let X i denote the yearly claim of policy holder i, i is equal to one, two, Etc , ten thousand
By Central Limit Theorem, X is equal to summation X i will have approximately Normal distribution with mean ten thousand into two hundred and sixty which is equal to two point six into ten to the power six and Standard deviation is eight hundred into square root of ten to the power four which is equal to eight into $^{\text{ten}}$ to the power four.

Hence, Probability of [X greater than two point eight into ten to the power six] is equal to Probability of [X minus two point six into ten to the power six by eight into $^{\text{ten}}$ to the power four greater than two point eight minus two point six into ten to the power six by eight into ten to the power four].
Which implies, Probability of [Z greater than twenty by eight] is equal to Probability of [Z greater than two point five] which is equal to zero point zero zero zero six two, From the table of Normal distribution,
That is, there are only six chances out of one thousand that the total yearly claim will exceed two point eight million dollars.

3. Illustrative Examples - Part 2

Problem 2:
In a game involving repeated throws of a balanced die, a person receives three rupees if the resulting number is greater than or equal to three and looses three rupees otherwise. Use central Limit Theorem to find the probability that in twenty five trials his total earnings exceed twenty five rupees.

Solution:

Figure 1

X_i	Probability (p)	X_i p	X_i^2 p
3	2/3	2	6
-3	1/3	-1	3
Total		1	9

Let X i be the earnings in the ith game. Then X takes values three and minus three with probabilities two by three and one by three respectively.
The product X i into p is obtained by multiplying Xi column and probability column. Similarly in the last column values of xi square into probabilities are obtained as six and three.
Expected value of X i is equal to summation X i into p which is equal to one. (That is, the total of third column).
Expected value of X i square is equal to summation X i square into p which is equal to nine. (That is, the total of fourth column).

Variance of X i is equal to Expected value of X i square minus expected value of X i whole square which is equal to eight. Let X be equal to X one plus X two plus etc till plus X twenty five. Then expected value of (X) is equal to n into Expected value of X i which is equal to twenty five into one which is equal to twenty five. Variance of X is equal to variance of summation X i which is equal to summation variance of X i which is equal to twenty five into eight equal to one hundred and twenty, by the Central Limit Theorem.

Z is equal to X minus twenty five by square root of one hundred and twenty follows Normal distribution with mean zero and variance one, asymptotically
Consider Probability of X greater than twenty five is equal to Probability of X minus twenty five by square root of one hundred and twenty is greater than zero is equal to Probability of Z greater than zero which is equal to zero point five, from the table of standard Normal probabilities.
Therefore the probability that the total earnings exceed twenty five rupees is equal to zero point five.

Problem 3:
The mean expenditure per customer at a tire store is eighty five dollars, with a standard deviation of nine dollars.
If a random sample of forty customers is taken, what is the probability that the sample average expenditure per customer for this sample will be eighty seven dollars or more?

Solution:
Since the sample size is greater than thirty, the central limit theorem says the sample means are normally distributed.
Z is equal to [X bar minus mu] by sigma by root n.
Z is equal to eighty seven minus eighty five by nine by root forty.
Z is equal to two by one point four two is equal to one point four one
For Z is equal to one point four one in the Z distribution table, the probability is zero point four two zero seven.

This represents the probability of getting a mean between eighty seven dollars and the population mean of eighty five dollars. Solving for the tail of the distribution yields, zero point five minus zero point four two zero seven is equal to zero point zero seven nine three. This is the probability of X-bar greater than or equal to eight seven dollars.

Interpretations:
Therefore, seven point nine three percent of the time, a random sample of forty customers from this population will yield a mean expenditure of eighty seven dollars or more.
OR,
From any random sample of forty customers, seven point nine three percent of them will spend on an average, eighty seven dollars or more.

4. Illustrative Examples - Part 3

Problem 4:
Let X i assume the values i and minus i with equal probabilities. Show that the law of large numbers cannot be applied to the independent variables X one, X two, etc

Solution:
We have Probability of X i equal to i is equal to one by two.
Probability of X i is equal to minus i is equal to one by two.
Expected value of X i is equal to summation x i into p of x i which is equal to zero.
Variance of X i is equal to Expected value of X i square minus expected value of X i whole square which is equal to i square.

Bn is equal to Variance of (X one plus X two plus etc till plus X n) is equal to variance of X one plus variance of X two plus etc till plus variance of X n which is equal to (one square plus two square plus etc till plus n square) which is equal to n into (n plus one) into (two n plus one) by six.
B n by n square tends to infinity as n tends to infinity.
The sufficient condition of Weak Law of Large Numbers is not satisfied by the given random variables. Hence Law of large numbers does not hold good.

Problem 5:
A scientist desires to estimate the mean of a population using a sample sufficiently large, such that, the probability will be zero point nine nine and that the sample mean will not differ from the population mean by more than twenty five percent of the standard deviation. How large should the sample size be?

Solution:
From the standard Normal distribution table, we have Probability of modulus of Z less than two point five eight is equal to zero point nine nine.
Therefore Probability of modulus of x bar minus mu by sigma by root n less than two point five eight is equal to zero point nine nine
Probability of modulus of x bar minus mu less than two point five eight into sigma by root n is equal to zero point nine nine.

But as per question, probability of modulus of x bar minus mu less than zero point two five sigma is equal to zero point nine nine.
Therefore, zero point two five sigma is equal to two point five eight sigma by root n.
Which implies n is equal to (two point five eight) square into sigma square by zero point two five into sigma whole square. Therefore n is equal to one hundred and six point five. Therefore sample size should be one hundred and seven.

5. Illustrative Examples - Part 4

Problem 6:
From a population with Standard deviation is equal to ten, a sample is drawn. Determine the size of the sample if the probability is at least zero point eight zero such that the sample mean will not differ from the population mean by more than two.

Solution:
To find n such that Probability of modulus of X bar minus mu less than two is greater than zero point eight zero.
Let x one, x two, etc x n be a random sample independently drawn. Let mean of the population be mu and sigma is equal to ten.
Let x one plus x two plus etc till plus x n be equal to X.

Since the sample is large , by Central limit theorem
X by n minus mu by sigma by root n follows Normal distribution with mean zero and variance one.
Therefore, probability of modulus of X bar minus mu less than two is greater than zero point eight zero. Which is equal to pprobability of modulus of X bar minus mu by sigma by root n less than two by sigma by root n is greater than zero point eight zero.

This implies probability of modulus of X bar minus mu by sigma by root n less than two by ten by root n is greater than zero point eight zero. Which is equal to, probability of modulus of Z less than two by ten by root n is greater than zero point eight zero.
This implies probability of modulus of Z less than zero point two by root n is greater than zero point eight zero. Which implies Probability of minus zero point two by root n less than Z less than zero point two root n is greater than zero point eight zero. Implies, integral from minus zero point two root n to zero point two root n of phi of Z d z is greater than zero point eight zero.

This implies twice the integral from zero to zero point two, square root n, phi of Z dz is greater than zero point eight zero.
Which implies the integral from zero to zero point two into square root of n, phi of Z dz is greater than zero point four zero.
From the standard normal distribution table, integral from zero to one point two eight of phi of Z dz is equal to zero point four zero.
Which implies zero point two square root of n is equal to one point two eight which implies n is equal to forty point nine six which implies n is equal to forty one.
Therefore n should be at least forty one.

Problem 7:
If x i's have only two values with equal probabilities i to the power alpha and (minus i to the power alpha), show that Law of Large Numbers can be applied to the independent variables x one, x two, etc till xn, only if alpha is less than one by two.

Solution:

Given X i takes only two values i to the power alpha and (minus i to the power alpha).
For the given problem, let probability of [X i is equal to i to the power alpha] be equal to probability of (x i is equal to (minus i to the power alpha) is equal to one by two.
Expected value of X i is equal to summation x i into p of x i which is equal to i to the power alpha into one by two plus (minus i to the power alpha) into one by two which is equal to zero.
Variance of X i is equal to Expected value of X i square minus expected value of X i whole square.
Expected value of X i square is equal to summation x i square into probability of x i which is equal to i to the power two alpha into one by two plus (minus i to the power two alpha) into one by two which is equal to i to the power two alpha.

Variance of X i is equal to Expected value of X i square minus expected value of X i whole square which is equal to i to the power two alpha.
Bn is equal to Variance of (X one plus X two plus etc till plus X n) which is equal to one to the power two alpha plus two to the power two alpha plus etc till n to the power two alpha which is equal to integral from zero to n of i to the power two alpha d i which is equal to i to the power two alpha plus one by two alpha plus one with limits zero to n.
B n is equal to n to the power two alpha plus one by two alpha plus one (From Euler Maclaurin's formula).

Limit of Bn by n square as n tends to infinity is equal to limit as n tends to infinity of n to the power two alpha plus one by two alpha plus one the whole into n square which is equal to limit as n tends to infinity of n to the power two alpha plus one minus two, divided by two alpha plus one which is equal to limit as n tends to infinity of n to the power two alpha minus one by two alpha plus one.
For Weak Law of Large numbers to hold good, limit as n tends to infinity of n to the power two alpha minus one by two alpha plus one should tend to 0 as n tends to infinity.
For which two alpha minus one should be less than zero which implies two alpha should be less than one implies alpha should be less than one by two
Hence the result.

Here's a summary of our learning in this session where we have:

- Understood the concept of Central Limit Theorem
- Understood the concept of Weak Law of Large numbers
- Described the basic concept of convergence criteria
- Demonstrated problems related to these theorems
- Described the procedure to apply Central Limit Theorem and Weak Law of Large numbers to the practical problems