Frequently Asked Questions

1. **What do you mean by Tchebyscheff's inequality?**

Answer:

Tchebysheff's inequality is one of the powerful tools that we can use in statistical analysis. Chebyshev allows us to estimate the deviation of the random variable from its mean. In Probability Theory **Chebyshev's inequality** (also spelled as **Tchebysheff's inequality**) guarantees that in any probability distribution ,"nearly all" values are close to the mean — the precise statement being that no more than $1/k^2$ of the distribution's values can be more than *k* standard deviations away from the mean.

In this inequality, we remove the restriction that the random variable has to be non negative. As a price, we now need to know additional information about the variable – (finite) expected value and (finite) variance.

Tchebyscheff's inequality Statement:

Let X be a random variable for which E(X) and V(X) exists. Then for any positive number k,

$$P\{|x-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

OR

$$P\{|x-\mu| \leq k\sigma\} \geq 1-\frac{1}{k^2}$$

2. **What do you understand by Markov's inequality?**

Answer:

Markov's inequality is the simplest concentration inequality. In probability theory, Markov's inequality gives an upper bound for the probability that a non-negative function of a random variable is greater than or equal to some positive constant.

random variable with finite expectation E(x) .Then for any $\varepsilon > 0$, P (X$\geq \varepsilon$) $\leq \frac{E(X)}{}$

Markov's inequality takes μ = E(X) into account and provides an upper bound on P{X ≥ ε } that depends on the values of ε and μ.

3. **A left-skewed distribution has a mean of 4.99 and a standard deviation of 3.13. Use** Chebyshev's Theorem **to calculate the proportion of observation you would**

expect to find within two standard deviations (in other words, between -2 and +2 standard deviations) from the mean:

Answer:

Given μ=4.99 and σ=2 and hence σ^2=4

By Tchebyscheff's The inequality $[P|X-\mu| \le k\sigma) \ge 1-\frac{1}{k^2}$

$$\Rightarrow P[-k\sigma \le (X-\mu) \le k\sigma] \ge 1-\frac{1}{k^2}$$

$$\Rightarrow P[-k*\sigma \le (X-4.99) \le k*\sigma] \ge 1-\frac{1}{k^2}$$

Let k=2. By substituting k=2 we get

$$\Rightarrow P[-2\sigma \le X-4.99 \le 2\sigma) \ge 1-\frac{1}{4} = 0.75$$

$$\Rightarrow [P|X-4.99| \le 2\sigma] \ge 0.75$$

At least **75%** of the observations fall between -2 and +2 standard deviations from the mean.

4. **Flip fair coin n times. Obtain a bound to obtain a probability of getting at least 70% of heads.**

Answer:

Let us flip fair coin n times. Let Xi be the indicator random variable for the event that the ith coin flip is head. Then X = ∑ Xi, is the number of heads in the sequence of n coin flips. Xi = 1 if the ith coin faces head.

= 0 otherwise

Since a fair coin is tossed, P (Xi=1) =1/2 for all i

Therefore,

E[X] = E (∑Xi) = ∑ E(xi)= ∑ P(Xi=1)=n/2

By Markov's inequality P $(X \ge \varepsilon) \le \frac{E(X)}{\varepsilon}$.

P $(X \ge 0.7n) \le \frac{E(X)}{0.7n} = \frac{n/2}{0.7n} = 0.714$.

The probability to obtain 70% or more heads in such a sequence of coin flips is thus bounded by 0.714

5. **X is a random variable with mean 10 and variance 9 .Find a lower bound to** $P\{|x-10| < 9\}$.

Answer:

By Tchebycheff's inequality

$$P\{|x-\mu| \le k\sigma\} \ge 1-\frac{1}{k^2}$$

Given μ=10 and σ^2 =9

Therefore

$$P\{|x-10| \le k.3\} \ge 1-\frac{1}{k^2}$$

Puttting 3k=9, k=3

$$1-\frac{1}{k^2} = 1-\frac{1}{9} = \frac{8}{9}$$

The lower bound of the probability $=1-\frac{1}{k^2} = \frac{8}{9}$ =0.89

6. For a Geometric distribution with probability function p(x)=2^{-x} ,x=1,2,…Prove that Tchebyscheff's inequality $[P|X-2| \le 2) \ge \frac{1}{2}$ while the actual probability is 15/16

Answer:

According to Tchebycheff's inequality

$$P\{|x-\mu| \le k\sigma\} \ge 1-\frac{1}{k^2}$$

p(x)= 2-x , x =1,2,....

Then E(X)= ∑xp(x) =∑x 2-x =1. ½ + 2 . (1/2)2 +3. (1/2)3 +.....=2

Similarly E(X2)= ∑x2p(x) =∑x2 2-x =12. ½ + 22 . (1/2)2 +32. (1/2)3 +.....=6

V(X) = E(X2) – [E(X)]2 = 6-(2)2 = 2

Therefore

$$P\{|x-\mu| \le C\} \ge 1-\frac{\sigma^2}{C^2}$$

$$\Rightarrow P\{|x-2| \le C\} \ge 1-\frac{2}{C^2} \quad \text{----(1)}$$

We need

$$[P|X-2| \le 2) \ge \frac{1}{2}$$

So let us put C=2 in equation (1), then we get

$$P\{|x-2| \le 2\} \ge 1-\frac{2}{2^2} = \frac{1}{2}$$

Now $P\{|x-2| \le 2\}$ = P[-2 ≤ X-2 ≤2] = P[0 < X ≤ 4]=p(1)+p(2)+p(3)+p(4)

=1/2+1/4+1/8+1/16=15/16

Hence it is proved that Tchebyschef's inequality $[P|X-2|\leq 2)\geq\frac{1}{2}$ while the actual probability is 15/16

7. Two unbiased dice are thrown. If X is the sum of the numbers showing up Then prove that $[P|X-7|\geq 3)<\frac{35}{54}$. Compare this with the actual probability

Answer:

Let X: the sum of the numbers showing up then

x	2	3	4	5	6	7	8	9	10
P(x)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36
x	11	12							
P(x)	2/36	1/36							

Then E(X) = ∑xp(x) = (1/36) [2+6+12+20+30+42+40+36+30+22+12]=7

E(X2) = ∑x2p(x) = 54.833

V(X) = E(X2) – [E(X)]2 =54.833- (7) 2 = 5.833

Applying Tchebycheff's inequality $[P|X-\mu|\geq C)<\frac{\sigma^2}{C^2}$

$$[P|X-7|\geq C)<\frac{5.833}{C^2}$$

Here C=3

$$[P|X-7|\geq 3)<\frac{5.833}{9}=\frac{35}{54}$$

To find the actual probability

$$[P|X-7|\geq 3)=P[(X-7)\succ 3or-(X-7)\geq 3]$$

$$=P(X\geq 10)+P(X<4)$$

$$=P(10)+P(11)+P(12)+P(2)+P(3)$$

=3/36+2/36+1/36+1/36+2/36=9/36=1/4

Hence the actual probability is 1/4

8. **X is a random variable with mean 5 and variance 4 find a lower bound to P{|x-5|<6}**

Answer:
By Tchebycheff's inequality

$$P\{|x-\mu| \le k\sigma\} \ge 1-\frac{1}{k^2}$$

Given μ=5 and σ^2 =4
Therefore

$$P\{|x-5| \le k.2\} \ge 1-\frac{1}{k^2}$$

Putting 2k=6, k=3

$$1-\frac{1}{k^2} = 1-\frac{1}{9}=\frac{8}{9}$$

The lower bound of the probability = $1-\frac{1}{k^2} = \frac{8}{9}$

9. **X is a random variable with mean 5 and variance 4. Find a upper bound to P{|x-5|≥6}**

Solution:

By Tchebycheff's inequality

$$P\{|x-\mu| \ge k\sigma\} \le \frac{1}{k^2}$$

Given μ=5 and σ^2 =4

Therefore

$$P\{|x-5| \ge k.2\} \le \frac{1}{k^2}$$

Puttting 2k=6, k=3

$$\frac{1}{k^2} = \frac{1}{9}$$

The upper bound of the probability = $\frac{1}{9}$

10. **X is a random variable with mean 2 .Find a upper bound to p { x > 8}**

Answer:

Given μ=2

By Markov's inequality P ($X \geq \varepsilon) \leq \dfrac{E(X)}{\varepsilon}$.

Therefore

P (X> 8) $\leq \dfrac{2}{8} = 0.25$

Hence the upper bound to $P\{X > 8\}$ is 0.25

11. If X is a random variable with mean =6 . Find t such that P {x ≥ t} ≤ 0.96

Answer:

Given μ=6 .

Therefore

By Markov's inequality P ($X \geq \varepsilon) \leq \dfrac{E(X)}{\varepsilon}$.

Substituting

P ($X \geq \varepsilon) \leq \dfrac{6}{\varepsilon}$.----(1)

Put $\dfrac{6}{\varepsilon} = 0.96 \Rightarrow \varepsilon = \dfrac{6}{0.96} = 6.25$

Substituting in (1)

P ($X \geq 6.25) \leq \dfrac{6}{\varepsilon}$ =0.96

Therefore t= 6.25

12. X is a random variable with mean 10 and variance 9 .Find a lower bound to P{ |x – 10 | < 9 }.

Answer:

Solution:

By Tchebycheff's inequality

$$P\{|x - \mu| \leq k\sigma\} \geq 1 - \frac{1}{k^2}$$

Given μ=10 and σ^2 =9

Therefore

$$P\{|x-10| \le k.3\} \ge 1-\frac{1}{k^2}$$

Puttting 3k=9, k=3

$$1-\frac{1}{k^2} = 1-\frac{1}{9} = \frac{8}{9}$$

The lower bound of the probability $= 1-\frac{1}{k^2} = \frac{8}{9}$ =0.89

13. If X is a random variable with mean =6 and variance =2 find t such that $P\{|x-6| < t\} \ge 0.96$

Answer:

Given μ=6 and σ^2 =2

Therefore

By Tchebycheff's inequality

$$P\{|x-\mu| \le k\sigma\} \ge 1-\frac{1}{k^2}$$

Substituting

$$P\{|x-6| \le k.\sqrt{2}\} \ge 1-\frac{1}{k^2} \text{ ----------------------(1)}$$

Put $1-\frac{1}{k^2}$ =0.96

Then $\frac{1}{k^2}$ =1-0.96=0.04 which implies k =5

Substituting in (1)

$$P\{|x-6| \le 5.\sqrt{2}\} \ge 0.96$$

But given that

$$P\{|x-6| < t.\} \ge 0.96$$

Therefore t= 5 √2 = 7.07

14. Let X be a Random variable taking values -1,0,1 with probabilities 1/8,6/8 and 1/8 respectively. Find using Tschebyscheff's inequality the upper bound of the probability $P\{|x| \ge 1\}$. **Compare this with the actual probability**

Answer:

Here X takes values -1, 0, and 1 with probabilities 1/8, 6/8 and 1/8 respectively
Then E(X) = ∑xp(x) = 0
V(X) = E(X^2) – [E(X)]^2 = ¼ -0 = ¼
Therefore

Given μ=0 and σ^2 =1/4

Applying Tchebycheff's inequality $[P|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$

Put μ=0 and σ=1/2 then $[P|X-0| \geq k\frac{1}{2}) \leq \frac{1}{k^2} \Rightarrow [P|X| \geq \frac{k}{2}) \leq \frac{1}{k^2}$

Put k/2 = 1 then k=2 Therefore 1/k^2 =1/4

Therefore $[P|X| \geq 1) \leq \frac{1}{4}$

The upper bound =1/4

To find the actual probability $|X| \geq 1$ means x ≥1 or x≤ -1

But X takes value only -1,0, 1. Therefore $|X| \geq 1$ means x =1 and x=-1

Therefore $P(|X| \geq 1) = P[x=1,-1] = P[x=1] + P[X=-1] = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

The upper bound is ¼ and the actual probability is also 1/4

15. For the number X obtained when a die is thrown prove that Tchebyscheff's inequality gives $[P|X-3.5| > 2.5) < 0.47$. Compare this with the actual probability

Answer:

p(x) = ½ for x = 1,2,3,..,6

Then E(X)= ∑xp(x) = (1X1/6)+(2X1/6)+……+(6X1/6)=3.5

V(X) = E(X2) – [E(X)]2 = 15.167-(3.5)2 = 2.917

Σ=√2.917=1.71

Applying Tchebycheff's $[P|X-\mu| > k\sigma) < \frac{1}{k^2}$ inequality

$[P|X-3.5| > k(1.71) < \frac{1}{k^2}$

Take k. (1.71) = 2.5 Then k = 2.5/1.71= 1.46

Therefore 1/k2 =0.47

Therefore $[P|X-3.5|>2.5)<0.47$

To find the actual probability $[P|X-3.5|>2.5)=1-P|X-3.5|\leq 2.5$

$$=1-P[-2.5+3.5\leq X\leq 2.5+3.5]=1-P[1\leq X\leq 6]=1-[\frac{1}{6}+\ldots+\frac{1}{6}]$$

$$=1-\frac{6}{6}=0$$