Distributions of functions of random variables

In this session we will discuss the problems of finding the probability distributions or densities of functions of one or more random variables. The entire session is divided in to the following subdivisions:

1. Introduction
2. Moment generating function technique
3. Distribution function technique
4. Transformation technique (One dimensional random variable**)**
5. Transformation technique (two dimensional random variable)
6. Conclusion

1. Introduction

Given a set of random variables $X_1,X_2,...X_N$ and their joint probability distributions or density, we shall be interested in finding the probability distribution or densities of some functions of random variables $Y=u(X_1,X_2,...X_N)$. i.e. the value of Y are related to those of the X's by means of the equation $y=u(x_1,x_2,...x_n)$.

Several methods are available for solving such problems. Some of the techniques which we are going to discuss are

1. Moment generating function technique
2. Distribution function technique
3. Transformation technique

2. **Moment generating function technique**: In situations where the function is a linear combination of n independent random variables $X_1,X_2,...X_N$, the moment generating function technique gives the simple solution.

Result: Let $X_1,X_2,... X_N$ are independent random variables and $Y = X_1+X_2+...+X_n$, then

$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$ where $M_{X_i}(t)$ is the value of the moment generating function of X_i at t.

E**xample 1**: Let X_1 and X_2 two independent Poisson random variables with respective parameters λ_1 and λ_2 Find the probability distribution of $X_1 + X_2$

Solution: Given X_1 and X_2 two independent Poisson random variables with respective parameters λ_1 and λ_2.

Then the mgf of X_1 is $M_{X_1}(t) = e^{\lambda_1(e^t-1)}$, and the mgf of X_2 is $M_{X_2}(t) = e^{\lambda_2(e^t-1)}$. Then from the properties of mgf, $M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$, the mgf of $X_1 + X_2$ is given by

$M_{X_1+X_2}(t) = M_{X_1}(t).M_{X_2}(t) = e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)} = e^{\lambda^*(e^t-1)}$ which is mgf of a Poisson variate with parameter $\lambda^* = \lambda_1 + \lambda_2$. Hence from uniqueness theorem of mgf,s $X_1 + X_2$ also a Poisson random variable with parameter $\lambda_1 + \lambda_2$

3. **Distribution function technique:**

In this technique we first find its distribution function and then its probability density by differentiation, Thus if $X_1, X_2, \ldots X_N$ are continuous random variables with a given joint probability density, the probability density of $Y=u(X_1, X_2, \ldots X_N)$ is first obtained by determining an expression for the probability distribution function $G(y) = P(Y \le y) = P[u(X_1, X_2, \ldots X_N) \le y]$ and then differentiating to get probability density g(y) , given by $g(y) = \frac{dG(y)}{dy}$

Example 2: The probability density of X is given by

$$f(x) = \begin{cases} 6x(1-x) & for\ 0 < x < 1 \\ 0 & elsewhere \end{cases}$$

, find the probability density of $Y=X^3$.

Solution: Let G(y) denote the value of the distribution function of Y at y, we can write

$G(y) = P(Y \le y) = P(X^3 \le y) = P(X \le y^{1/3}) = \int_0^{y^{1/3}} 6x(1-x)dx = 3y^{2/3} - 2y$

hence the pdf of Y is $g(y) = 2(y^{-1/3} - 1)$ for $0<y<1$.

o elsewhere

Example 3. : Find the probability density of Y=|X| when X is the standard normal distribution.

Solution: Let X is a continuous random variable with f(x) is the value of the probability density of X at Z=x and F(x) be its Distribution function at x. Let Y=|X|.

denote G(y) the value of the distribution function of Y at y, we can write

G(y) = P(Y≤y) = P(|X|≤y) = P(-y<X<y)= F(y) -F(-y)

Differentiation we get the density function of Y as g(y) = f(y)+f(-y) for all y>0. (*)

Given X is standard normal variate, and its density function is given by f(x) = $\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}x^2)$ replace x by y we get the value of f(y) = $\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}y^2)$ and replacing x by -y, we get the value of f(-y) = $\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}y^2)$ substituting in * we get the density of Y= |X| as

g(y) =

$2\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}y^2)$ = $\sqrt{\frac{2}{\pi}}\exp(-\frac{1}{2}y^2)$, y>0

Example 4: The joint density of X_1 and X_2 is given by f(x_1, x_2) = $\begin{cases} 6e^{-3x1-2x2} & for\ x_1 > 0, x_2 > 0 \\ 0 & elsewhere \end{cases}$

find the probability density of Y= X_1+X_2

Solution:

The distribution function of Y = X_1+X_2 is given by

$G(y) = \int_0^y \int_0^{y-x_2} 6e^{-3x1-2x2}dx_1dx_2$ = 1+2e^{-3y}-3e^{-2y}

and differentiating with respect to y we get the density of Y as

g(y) = 6(e^{-2y}-e^{-3y}) for y>0 and

0 elsewhere

4. **Transformation technique (One dimensional random variable)**

Let f(x) be the value of the probability density of the continuous random variable X at x. If the function given by y=u(x) is differentiable and either increasing or decreasing for all values within the range of X for which f(x) $\neq 0$, then, for these values of x, the equation y=u(x) can be uniquely solved for x which gives x=w(y), and for the corresponding values of y the probability density of Y = u(X) is given by

$g(y) = f[w(y)].|w'(y)| \textit{provided } u'(x) \neq 0$, elsewhere g(y) = 0 (**)

Example 5: If X has exponential distribution given by

$$f(x) = \begin{cases} e^{-x} & \textit{for } x > 0 \\ 0 & \textit{elsewhere} \end{cases}$$

find the probability density of the random variable Y=√X

Solution: The probability density of the random variable X is

f(x) = e^{-x} for x>0.

Consider the transformation Y=√X, which has the unique inverse x=y^2

and $w'(y)$ = dx/dy = 2y.

Using ** The density function of transformed random variable Y is

$g(y) = f[w(y)].|w'(y)|$

=$e^{-y^2}2y$ for y>0.

Thus the probability density function of Y is given by

$$G(y) = \begin{cases} 2ye^{-y^2} & \textit{for } y > 0 \\ 0 & \textit{elsewhere} \end{cases}$$

Example 6: Let X has Uniform density in the range (0,1) , find the distribution of Y= e^{X}.

Solution:

The density function of X is $f(x) = 1, \quad 0 < x < 1$

Consider the transformation Y= $e^{X.}$ The inverse transformation is X= log Y. and $w'(y)$ = dx/dy = 1/y.

Thus the probability density function of Y is given by

g(y) = 1. 1/y, 0< logy<1

Or

g(y) = 1/y, 1< y<e,

zero elsewhere.

5. **Transformation technique (two dimensional random variable)**

let (X,Y) be bivariate random variable with joint density f(x,y) , transform the random variables X and Y to new random variables U and V by means of transformation u=u(x,y) and v=v(x,y) , where u and v are continuously differentiable functions. Define jacobian of the transformation as J= $\frac{\partial(x,y)}{\partial(u,v)}$ = $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix}$.

J is either >0 or <0 throughout the (x,y) plane so that the inverse transformation is uniquely given by x=x(u,v) and y= y(u,v) , and assume that all the 4 partial derivatives exists and continuous. Then the joint density function of the transformed random variables U and V is

g(u,v)= f(x(u,v), y(u,v)) |J| ***where |J| is the Modulus values of the Jacobian of the transformation and f(x,y) is expressed in terms of u and v.

Example 7:

Let (X,Y) have the joint density function f(x,y) = $\begin{cases} xe^{-x}e^{-y} & \text{for } x, y > 0 \\ 0 & \text{elsewhere} \end{cases}$

Find the probability distribution for the variables Z and U where Z = X+Y and U= Y/X

Solution: Let Z= X+Y and U=Y/X

The inverse transformation is $X = \frac{Z}{U+1}$ and $y = U\frac{Z}{U+1}$

The Jacobian of the transformation is $|J| = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial u} \end{vmatrix}$

$$\begin{vmatrix} \frac{1}{u+1} & \frac{-z}{(u+1)^2} \\ \frac{u}{u+1} & \frac{z}{(u+1)^2} \end{vmatrix} = \frac{z}{(u+1)^2}$$

Using *** the joint density of the transformed random variable (U,V) is given by

g(z,u)= f(x(z,u), y(z,u)) |J|

$$= \frac{z}{u+1} e^{-\frac{z}{u+1}} e^{-\frac{uz}{u+1}} \frac{z}{(u+1)^2}$$

$$= \frac{z^2}{(u+1)^3} e^{-z/(u+1)} e^{-uz/(u+1)} \qquad z,u>0$$

Or $g(z,u) = \frac{z^2}{(u+1)^3} e^{-z}$, z,u>0

If we wanted to know the density functions of Z and U separately, just integrate out the other variable in the normal way.

i.e. The density function of Z is given by $g(z) = \int_{-\infty}^{\infty} g(z,u)du = \int_{0}^{\infty} g(z,u)du$

$$= \int_0^{\infty} \frac{z^2}{(u+1)^3} e^{-z} du = \frac{z^2 e^{-z}}{2} \quad \text{for } z>0$$

And $g(u) = \int_{-\infty}^{\infty} g(z,u)dz$ = $\int_{0}^{\infty} \frac{z^2}{(u+1)^3} e^{-z} dz$ = $\frac{2}{(u+1)^3}$ for u>0

Example 8:

If joint probability density of X_1 and X_2 is given by

f(x1, x2) = $\begin{matrix} 1 & for\ 0 < x_1 < 1 and\ 0 < x_2 < 1 \\ 0 & elsewhere \end{matrix}$

(i) Find the probability density of Y = $X_1 + X_2$ and Z = X_2

(ii) Find the marginal density of Y

Solution:Consider the transformation

Y = $X_1 + X_2$ and Z =X_2 (1)

The inverse transformation is

X_1=Y-Z and X_2 = Z (2)

The jacobian of the transformation is

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

Hence $|J| = 1$

The joint pdf of Y and Z is

$g(y,z) = 1 \ . \ 1$ = 1

The range for Y and Z is

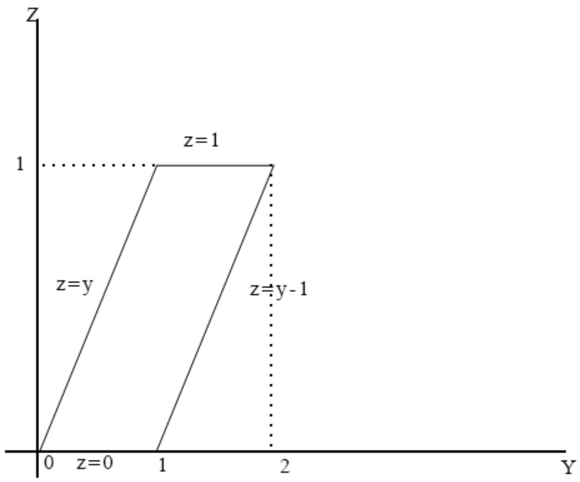
From (1) 0<y<2 and 0<z<1

From (2) 0<y-z<1 and or z<y<z+1

The joint density of Y and Z can be written as

$$g(y,z) = 1\ , \quad z < y < z+1, \quad 0 < z < 1$$
$$and \quad 0 \ \ elsewhere$$

The transformed sample space is



Integrating seperately for 0<y<1 and 1<y<2

We get the density of y as

$$g(y) = \begin{cases} o & y \le 0 \\ \int_{0}^{y} 1 \, dz & 0 < y < 1 \\ \int_{y-1}^{1} 1 \quad dz & 1 < y < 2 \\ 0 & y \ge 0 \end{cases} \qquad 1 \,.\quad 1$$

Or

$$g(y) = \begin{cases} o & for \quad y \le 0 \\ y & for \quad 0 < y < 1 \\ 2 - y & for \quad 1 < y < 2 \\ 0 & for \quad y \ge 0 \end{cases}$$

Quiz:

1. Following is the technique to find the probability distribution of functions of one or more random variables.

   a. Distribution function technique
   b. Transformation technique
   c. Moment Generating function Technique
   d. **All of the above**

2. Let Y = |X| and f(x) and g(y) are values of pdf of X at x and Y at y respectively, then the expression for g(y) is given by

   a. **f(y)+f(-y)**
   b. f(y)+f(y)
   c. f(y)-f(-y)
   d. can not wroite

3. . The moment generating function technique gives the simple solution to to find the probability distribution of functions of one or more random variables

when

   a. the function is a linear combination of independent random variables
   b. the function is a nonlinear combination of independent random variables
   c. the function is a nonlinear combination of dependent random variables
   d. the function is a linear combination of dependent random variables

4. : Let X_1 and X_2 two independent Poisson random variables with respective parameters λ_1 and λ_2. then the probability distribution of $X_1 + X_2$

   a. **Poisson random variable with parameter $\lambda_1 + \lambda_2$**
   b. Binomial random variable with parameter $\lambda_1 + \lambda_2$
   c. Geometric random variable with parameter $\lambda_1 + \lambda_2$
   d. Poisson random variable with parameter

5. Sum of n independent Bernoulli random variables with constant probability of success is a

   a. Bernoulli random variable
   b. **Binomial random variables**
   c. Poisson random variables
   d. None of the above

6. Let X follow Uniform distribution in the interval (0,1) , the distribution of Y== -2logX is

   a. Beta Random Variable
   b. **Gamma Random Variable**
   c. Uniform Random variable
   d. Cauchy Random Variable

7. The density function of X is $f(x) = 1, \quad 0 < x < 1$. In obtaining the distribution of Y= e^X , The jacobian of the transformation is

a. 1
b. y
c. **1/y**
d. e^{-y}

8. The joint probability density of X_1 and X_2 is given by

$$f(x1, x2) = \begin{matrix} e^{-(x1,x2)} & for\ x_1, x_2 > 0 \\ 0 & elsewhere \end{matrix}$$

The limit for the tranfsformed Random Variable $Y = \frac{X_1}{X_1 + X_2}$ is

a. **0<y<1**
b. 0<y<∞
c.-∞ < y<∞
d. 0<y<1/2

9. The joint probability density of X_1 and X_2 is given by

$$f(x1, x2) = \begin{matrix} e^{-(x1,x2)} & for\ x_1, x_2 > 0 \\ 0 & elsewhere \end{matrix}$$

To Obtain the joint density of $Y = \frac{X_1}{X_1 + X_2}$ and $Z = X_1+X_2$, , the jacobian of the transfromation is n

a. **-z**
b. +z
c. 1+z
d. 1-z

10. If joint probability density of X_1 and X_2 is given by

$$f(x1, x2) = \begin{matrix} 1 & for\ 0 < x_1 < 1\ and\ 0 < x_2 < 1 \\ 0 & elsewhere \end{matrix}$$, In sloving the joint probability density of $Y = X_1 + X_2$ and $Z = X_2$, the jacobian is

a. **1**
b. 2
c. 0
d. none of the above

11. If joint probability density of X_1 and X_2 is given by

f(x1, x2) = $\begin{matrix} 1 & for\ 0 < x_1 < 1\, and\ \ 0 < x_2 < 1 \\ 0 & elsewhere \end{matrix}$, the range for the trnsformed random variable Y = $X_1 + X_2$ is

a. 0<y<∞
b. 0<y<1
c. -∞<y<∞
d. **0<y<2**

12 Let f(x) be the value of the probability density of the continuous random variable X at x. In deriving the probability function of transformed random variable y=u(x), the inverse function should be

. a. **Monotonic**
b. bell shaped
c. discontinuous
d. none of the above

13. If X has exponential distribution given by , f(x) = $e^{-x} \quad for\ \ x > 0$, the probability density of the random variable Y=√X

a. $2ye^{-y^2}$
b. $2y^2e^{-y^2}$
c. e^{-y^2}
d. ye^{-y^2}

14. The density function of X is $f(x) = 1, \quad 0 < x < 1$, then the density of Y= e^{X}. is

a. y for 1< y<e,
b. 1 for 1< y<e,
c. **1/y for 1< y<e,**
d. none of the above

15 I f F(x) is the value of the distribution function of the continuous random variable X at x then the probability density of Y=F(x) is
a. Uniform (0,1)
b. Uniform (a, b)
c. Normal (0,1)

d. Depends on the distribution of X

Frequently asked questions:

1. **What all the techniques available to find the distributions or densities of functions of one or more random variables?**

   Commonly used techniques to find the distributions or densities of functions of one or more random variables are

   a. Moment generating function technique
   b. Distribution function technique
   c. Transformation technique

2. **Describe Moment generating function technique to find the distributions of functions of random variables**:

   In situations where the function is a linear combination of n independent random variables $X_1, X_2, \ldots X_N$, the moment generating function technique gives the simple solution.
   Let $X_1, X_2, \ldots X_N$ are independent random variables and $Y = X_1 + X_2 + \ldots + X_n$, then

$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$ where $M_{X_i}(t)$ is the value of the moment generating function of X_i at t.

Identify the probability distribution corresponding to $M_Y(t)$, which gives the probability distribution of Y

3. **Let X_1 and X_2 two independent Poisson random variables with respective parameters λ_1 and λ_2 Find the probability distribution of $X_1 + X_2$**

Solution: Given X_1 and X_2 two independent Poisson random variables with respective parameters λ_1 and λ_2.

Then the mgf of X_1 is $M_{X_1}(t) = e^{\lambda_1(e^t - 1)}$, and the mgf of X_2 is $M_{X_2}(t) = e^{\lambda_2(e^t - 1)}$. Then from the properties of mgf, $M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$, the mgf of $X_1 + X_2$ is given by

$M_{X_1+X_2}(t) = M_{X_1}(t).M_{X_2}(t) = e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)} = e^{\lambda^*(e^t-1)}$ which is mgf of a Poisson variate with parameter $\lambda^* = \lambda_1 + \lambda_2$. Hence from uniqueness theorem of mgf,s $X_1 + X_2$ also a Poisson random variable with parameter $\lambda_1 + \lambda_2$

4. **Describe Distribution function technique to find the distributions of functions of random variables :**

In this technique we first find its distribution function and then its probability density by differentiation, Thus if $X_1, X_2, ... X_N$ are continuous random variables with a given joint probability density, the probability density of $Y = u(X_1, X_2, ... X_N)$ is first obtained by determining an expression for the probability distribution function $G(y) = P(Y \le y) = P[u(X_1, X_2, ... X_N) \le y]$ and then differentiating to get probability density g(y) , given by $g(y) = \frac{dG(y)}{dy}$

4. **The probability density of X is given by**

$$f(x) = \begin{cases} 6x(1-x) & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$ **, find the probability density of $Y=X^3$.**

Solution: Let G(y) denote the value of the distribution function of Y at y, we can write

$$G(y) = P(Y \le y) = P(X^3 \le y) = P(X \le y^{1/3}) = \int_0^{y^{1/3}} 6x(1-x)\,dx = 3y^{2/3} - 2y$$

hence the pdf of Y is $g(y) = 2(y^{-1/3} - 1)$ for $0<y<1$.

o elsewhere

5. . : Find the probability density of $Y=|X|$ when X is the standard normal distribution.

Solution: Let X is a continuous random variable with f(x) is the value of the probability density of X at Z=x and F(x) be its Distribution function at x. Let $Y=|X|$.

denote G(y) the value of the distribution function of Y at y, we can write

$G(y) = P(Y \le y) = P(|X| \le y) = P(-y<X<y) = F(y) - F(-y)$

Differentiation we get the density function of Y as $g(y) = f(y) + f(-y)$ for all $y>0$. (*)

Given X is standard normal variate, and its density function is given by f(x) = $\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}x^2)$ replace x by y we get the value of f(y) = $\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}y^2)$ and replacing x by -y, we get the value of f(-y) = $\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}y^2)$ substituting in * we get the density of Y= |X| as g(y) =

$2\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}y^2)$ = $\sqrt{\frac{2}{\pi}}\exp(-\frac{1}{2}y^2)$, y>0

6. The joint density of X_1 and X_2 is given by f(x_1, x_2) = $\begin{matrix} 6e^{-3x1-2x2} & \textit{for } x_1 > 0, x_2 > 0 \\ 0 & \textit{elsewhere} \end{matrix}$ **find the probability density of Y= X_1+X_2**

Solution:

The distribution function of Y = X_1+X_2 is given by

$G(y) = \int_0^y \int_0^{y-x_2} 6e^{-3x1-2x2} dx_1 dx_2$ = 1+2e^{-3y}-3e^{-2y}

and differentiating with respect to y we get the density of Y as

g(y) = 6(e^{-2y}-e^{-3y}) for y>0 and

0 elsewhere

7. **Explain Transformation technique find the distributions of functions of random variables** in **One dimensional random variable case.**

Let f(x) be the value of the probability density of the continuous random variable X at x. If the function given by y=u(x) is differentiable and either increasing or decreasing for all values within the range of X for which f(x) ≠0, then, for these values of x, the equation y=u(x) can be uniquely solved for x which gives x=w(y), and for the corresponding values of y the probability density of Y = u(X) is given by

$g(y) = f[w(y)].|w'(y)| \textit{provided } u'(x) \neq 0$, elsewhere g(y) = 0 (**)

8. If X has exponential distribution given by

$$f(x) = \begin{cases} e^{-x} & for\ \ x > 0 \\ 0 & elsewhere \end{cases}$$

find the probability density of the random variable Y=√X using transformation technique.

Solution: The probability density of the random variable X is

$f(x) = e^{-x}$ for x>0.

Consider the transformation Y=√X, which has the unique inverse $x=y^2$

and $w'(y)$ = dx/dy = 2y.

Using ** The density function of transformed random variable Y is

$$g(y) = f[w(y)].|w'(y)|$$

$=e^{-y^2}2y$ for y>0.

Thus the probability density function of Y is given by

$$G(y) = \begin{cases} 2ye^{-y^2} & for\ \ y > 0 \\ 0 & elsewhere \end{cases}$$

9.: **Let X has Uniform density in the range (0,1) , find the distribution of Y= $e^{X.}$**

Solution:

The density function of X is $f(x) = 1, \quad 0 < x < 1$

Consider the transformation Y= $e^{X.}$ The inverse transformation is X= log Y. and $w'(y)$ = dx/dy = 1/y.

Thus the probability density function of Y is given by

g(y) = 1. 1/y, 0< logy<1

Or

g(y) = 1/y, 1< y<e,

zero elsewhere.

10. **Explain Transformation technique find the distributions of functions of random variables in two dimensional random variable case**

let (X,Y) be bivariate random variable with joint density f(x,y) , transform the random variables X and Y to new random variables U and V by means of transformation u=u(x,y) and v=v(x,y) , where u and v are continuously differentiable functions. Define jacobian of the transformation

as $J=\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix}$.

J is either >0 or <0 throughout the (x,y) plane so that the inverse transformation is uniquely given by x=x(u,v) and y= y(u,v) , and assume that all the 4 partial derivatives exists and continuous. Then the joint density function of the transformed random variables U and V is

g(u,v)= f(x(u,v), y(u,v)) |J| ***where |J| is the Modulus values of the Jacobian of the transformation and f(x,y) is expressed in terms of u and v.

11. Let (X,Y) have the joint density function f(x,y) = $\begin{matrix} xe^{-x}e^{-y} & for\ x, y > 0 \\ 0 & elsewhere \end{matrix}$

Find the probability distribution for the variables Z and U where Z = X+Y and U= Y/X

Solution: Let Z= X+Y and U=Y/X

The inverse transformation is X = $\frac{Z}{U+1}$ and y = $U\frac{Z}{U+1}$

The Jacobian of the transformation is |J| = $\begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial u} \end{vmatrix}$

$$\begin{vmatrix} \frac{1}{u+1} & \frac{-z}{(u+1)^2} \\ \frac{u}{u+1} & \frac{z}{(u+1)^2} \end{vmatrix} = \frac{z}{(u+1)^2}$$

Using *** the joint density of the transformed random variable (U,V) is given by

g(z,u)= f(x(z,u), y(z,u)) |J|

$$= \frac{z}{u+1} e^{-\frac{z}{u+1}} e^{-\frac{uz}{u+1}} \frac{z}{(u+1)^2}$$

$$= \frac{z^2}{(u+1)^3} e^{-z/(u+1)} e^{-uz/(u+1)} \qquad z,u>0$$

Or g(z,u) = $\frac{z^2}{(u+1)^3} e^{-z}$, z,u>0

If we wanted to know the density functions of Z and U separately, just integrate out the other variable in the normal way.

i.e. The density function of Z is given by $g(z) = \int_{-\infty}^{\infty} g(z,u)du = \int_{0}^{\infty} g(z,u)du$

$$= \int_0^\infty \frac{z^2}{(u+1)^3} e^{-z} du = \frac{z^2 e^{-z}}{2} \text{ for } z>0$$

And $g(u) = \int_{-\infty}^{\infty} g(z,u)dz = \int_0^\infty \frac{z^2}{(u+1)^3} e^{-z} dz = \frac{2}{(u+1)^3}$ for u>0

12. **The joint probability density of X_1 and X_2 is given by**

f(x1, x2) = $\begin{cases} e^{-(x1,x2)} & \text{for } x_1, x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$

Find the probability density of Y = $\dfrac{X_1}{X_1 + X_2}$

Solution: Consider the transformation

Y = $\dfrac{X_1}{X_1 + X_2}$ and $Z = X_1+X_2$

The inverse transformation is

$X_1=ZY$ and $X_2= Z-ZY = Z(1-Y)$

The jacobian of the transformation is

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial z} & \dfrac{\partial x_1}{\partial y} \\ \dfrac{\partial x_2}{\partial z} & \dfrac{\partial x_2}{\partial y} \end{vmatrix} = \begin{vmatrix} y & z \\ 1-y & -z \end{vmatrix} = -z$$

Hence $|J| = z$

The joint pdf of Z and Y is

$$g(z, y) = e^{-z} z \quad \text{for} \quad 0 < y < 1,\ 0 < z < \infty$$

Integrating the jiont density g(z,y) with respect to z we get the density of y as

$$g(y) = \int_0^\infty z e^{-z}\ dz$$

$= \Gamma 2$

= 1 for $0<y<1$ and 0 elsewhere

13. If joint probability density of X_1 and X_2 is given by

f(x1, x2) = $\begin{cases} 1 & for\ 0 < x_1 < 1 \, and\ \ 0 < x_2 < 1 \\ 0 & elsewhere \end{cases}$

(iii) **Find the probability density of Y =** $X_1 + X_2$ **and Z =** X_2

(iv) **Find the marginal density of Y**

Solution:Consider the transformation

Y = $X_1 + X_2$ and Z =X_2 (1)

The inverse transformation is

X_1=Y-Z and X_2 = Z (2)

The jacobian of the transformation is

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

Hence $|J| = 1$

The joint pdf of Y and Z is

$g(y, z) = 1 \ . \ \ 1$ = 1

The range for Y and Z is

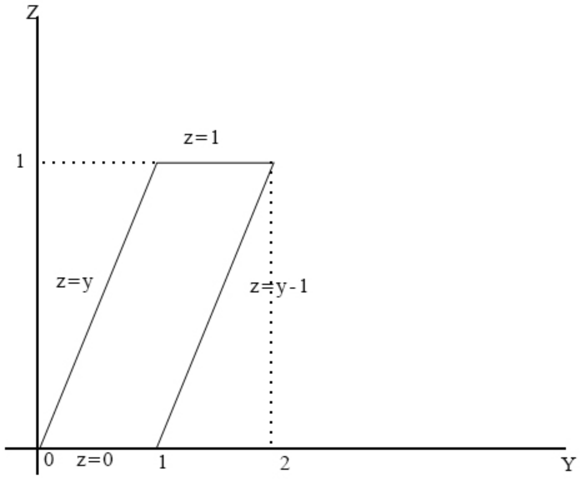
From (1) $0<y<2$ and $0<z<1$

From (2) $0<y-z<1$ and or $z<y<z+1$

The joint density of Y and Z can be written as

$$g(y, z) = 1 \ , \quad z < y < z + 1, \quad 0 < z < 1$$
$$and \quad 0 \ \ elsewhere$$

The transformed sample space is



Integrating seperately for $0<y<1$ and $1<y<2$

We get the density of y as

$$g(y) = \begin{cases} o & y \le 0 \\ \int_{0}^{y} 1 \, dz & 0 < y < 1 \\ \int_{y-1}^{1} 1 \quad dz & 1 < y < 2 \\ 0 & y \ge 0 \end{cases} \qquad 1 \,. \quad 1$$

Or

$$g(y) = \begin{cases} o & for \quad y \le 0 \\ y & for \quad 0 < y < 1 \\ 2 - y & for \quad 1 < y < 2 \\ 0 & for \quad y \ge 0 \end{cases}$$

Conclusion: In this session we have discussed the problem of finding the probability distributions or densities of one or more random variables. Severl methods are available for solving this kind of problems. We have discussed the three methods : Moment generating function technique, Distribution function technique and Transformation technique to solve such problems. Although all three methods can be used in some situations, in most problem one technique will be preferable, which is easier to use than the other. We have discussed only the continuous case as in discrete case there is no real problem as long as the relationship between the variables of X and Y = U(X) is one to one, hence we can solve the problem by taking appropriate substitution.