Snedecor's F- Distribution

In this session we discuss Snedecor's F- Distribution and its properties. The entire session is divided into following subdivisions.

1. Derivation of F distribution
2. Mean of the distribution
3. Other moments of the distribution
4. Mode of the distribution
5. Special cases
6. Some results
7. Applications
8. Conclusion

1. Derivation of F distribution

An distribution that plays an important role in connection with sampling from normal population is the F distribution, named after Sir, Ronald A Fisher, one of the most prominent statistician . F distribution is studied as the sampling distribution of the ratio of two independent random variables with chi-square distributions, each divided by its respective degree of freedom.

Theorem: Let U and V are two independent random variables having chi-square distribution with v1 and v2 degree of freedom, then X= $\frac{U/v_1}{V/v_2}$ is a random variable having F distribution whose probability density is given by

$$f(x) = \frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B\left(\frac{v1}{2}, \frac{v2}{2}\right)} \cdot \frac{x^{\frac{v1}{2}-1}}{\left(1+\frac{v1}{v2}x\right)^{(v1+v2)/2}}, \quad 0 < x < \infty \quad \text{.......(1)}$$

Proof: Let U and V are independent random variables having chi-square distribution with v1 and v2 degree of freedom.

Then the joint density of U and V is given by

g(u,v) = g(u) . g(v)

$$= \frac{1}{2^{v1/2}\Gamma(v1/2)} u^{\frac{v1}{2}-1} e^{-u/2} \frac{1}{2^{v2/2}\Gamma(v2/2)} v^{\frac{v2}{2}-1} e^{-v/2} \text{ ,u>0, v>0}$$

Or $$\frac{1}{2^{(v1+v2)/2}\Gamma(v1/2)}\frac{1}{\Gamma(v2/2)}u^{\frac{v1}{2}-1}v^{\frac{v2}{2}-1}e^{-(u+v)/2} \quad u>0, v>0$$

Consider the transformation

X= $\frac{U/v_1}{V/v_2}$ and Y=V

The inverse transformation is

U= $XY\frac{v_1}{v_2}$ V=Y

The jacobian of the transformation is

J = $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ = $\begin{vmatrix} \frac{yv1}{v2} & \frac{xv1}{v2} \\ 0 & 1 \end{vmatrix}$ = y v_1/v_2

Hence the joint density of X and Y is

f(x,y) = $$\frac{1}{2^{(v1+v2)/2}\Gamma(v1/2)}\frac{1}{\Gamma(v2/2)}(xy\frac{v1}{v2})^{\frac{v1}{2}-1}y^{\frac{v2}{2}-1}e^{-(xy\frac{v1}{v2}+y)/2}y\frac{v_1}{v_2}$$

= $$\frac{(\frac{v1}{v2})^{v1/2}}{2^{(v1+v2)/2}\Gamma(v1/2)}\frac{1}{\Gamma(v2/2)}(x)^{\frac{v1}{2}-1}y^{\frac{v1}{2}+\frac{v2}{2}-1}e^{-y/2(1+x\frac{v1}{v2})}$$ for x>0 and y>0

Now, integrating out with respect to y we get the distribution of X

I.e. f(x) = $\int_0^\infty f(x,y)dy$

$$= \frac{(\frac{v1}{v2})^{v1/2}}{2^{(v1+v2)/2}\Gamma(v1/2)} \frac{1}{\Gamma(v2/2)}(x)^{\frac{v1}{2}-1}\int_0^\infty y^{\frac{v1}{2}+\frac{v2}{2}-1} e^{-y/2(1+x\frac{v1}{v2})} dy$$

By making the substitution w=($y/2(1+x\frac{v1}{v2}$) we get

$$f(x) = \frac{(\frac{v1}{v2})^{v1/2}}{2^{(v1+v2)/2}\Gamma(v1/2)} \frac{1}{\Gamma(v2/2)}(x)^{\frac{v1}{2}-1}\int_0^\infty \left(\frac{2w}{1+x\frac{v1}{v2}}\right)^{(\frac{v1+v2}{2}-1)} e^{-w}\frac{2}{1+x\frac{v1}{v2}}dw$$

$$= \frac{(\frac{v1}{v2})^{v1/2}}{2^{(v1+v2)/2}\Gamma(v1/2)} \frac{1}{\Gamma(v2/2)}(x)^{\frac{v1}{2}-1}\left(\frac{2}{1+x\frac{v1}{v2}}\right)^{(\frac{v1+v2}{2})}\int_0^\infty w^{(\frac{v1+v2}{2}-1)} e^{-w}dw$$

$$= \frac{(\frac{v1}{v2})^{v1/2}}{2^{(v1+v2)/2}\Gamma(v1/2)} \frac{1}{\Gamma(v2/2)}(x)^{\frac{v1}{2}-1}\left(\frac{2}{1+x\frac{v1}{v2}}\right)^{(\frac{v1+v2}{2})}\Gamma\left(\frac{v1+v2}{2}\right)$$

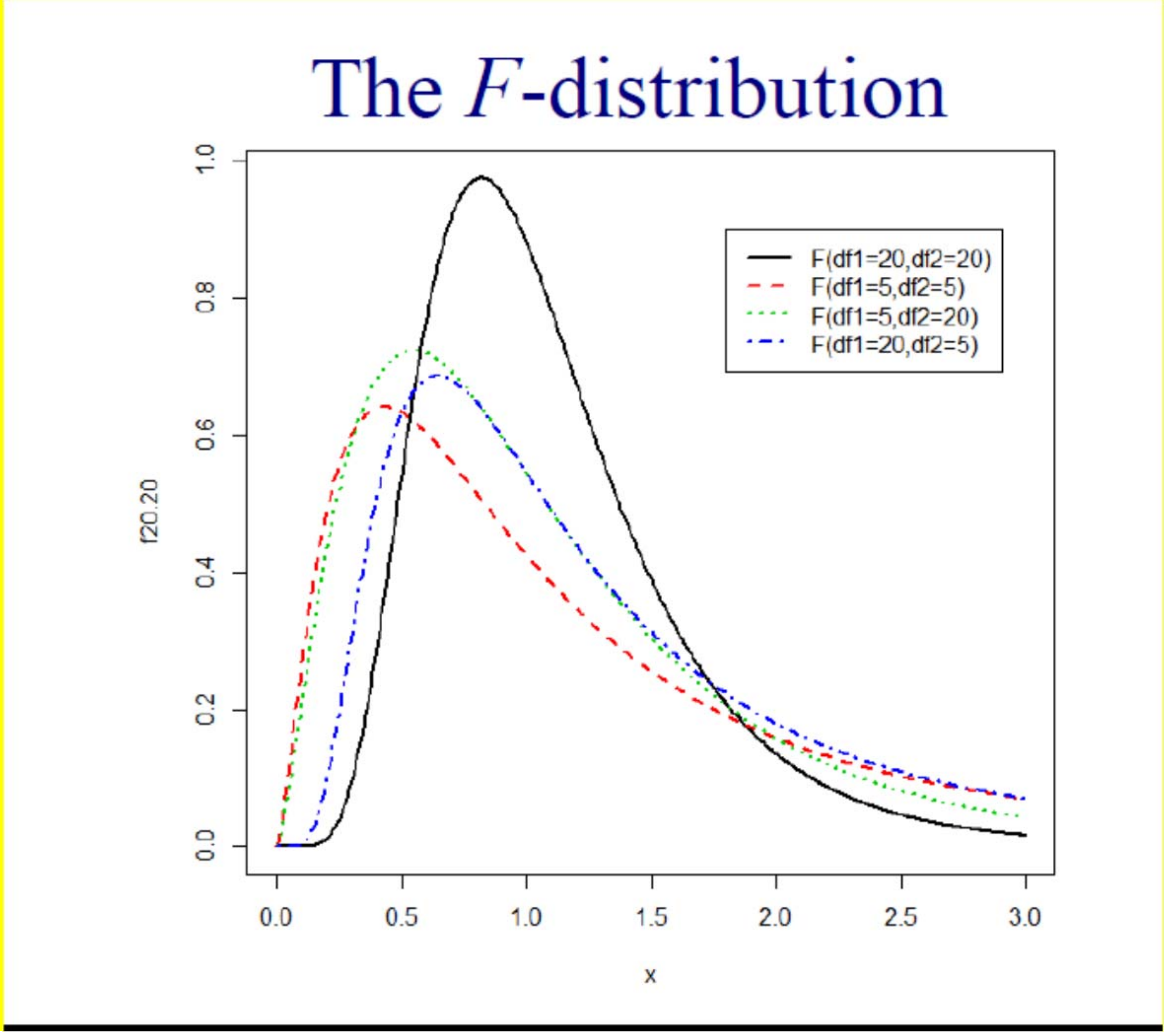
Or

$$f(x) = \frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})}\cdot\frac{x^{\frac{v1}{2}-1}}{\left(1+\frac{v1}{v2}x\right)^{(v1+v2)/2}},\quad 0 \le x < \infty$$

Using symbol we can write as X~F(v1, v2)

F-distribution probability curve :

F-distributions are generally skewed. The shape of an *F*-distribution depends on the values of *v1* and *v2* the degrees of freedoms



2. **Mean of the distribution**

Mean = E(X) = $\int_0^\infty x \; f(x)dx$

$$= \int_0^\infty x \frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})} \cdot \frac{x^{\frac{v1}{2}-1}}{\left(1+\frac{v1}{v2}x\right)^{(v1+v2)/2}} dx$$

$$= \frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})} \int_0^\infty x. \frac{x^{\frac{v1}{2}-1}}{\left(1+\frac{v1}{v2}x\right)^{(v1+v2)/2}} dx$$

By making the substitution 1/y=($1+\frac{v1}{v2}x$) we get or x= $\frac{1-y}{y}\frac{v2}{v1}$

$$E(X) = \frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})} \int_0^1 . \frac{\left(\frac{1-y}{y}\frac{v2}{v1}\right)^{\frac{v1}{2}}}{(1/y)^{(v1+v2)/2}} (1/y^2)\frac{v2}{v1} dy$$

$$= \frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})} \left(\frac{v2}{v1}\right)^{(v1/2+1)} \int_0^1 y^{v2/2-2}.(1-y)^{v1/2} dy$$

$$= \frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})} \left(\frac{v2}{v1}\right)^{(v1/2+1)} B(\tfrac{v1}{2}+1,\tfrac{v2}{2}-1)$$ as $$B(\tfrac{v1}{2}+1,\tfrac{v2}{2}-1)$$ =

$$\int_0^1 y^{v2/2-2}.(1-y)^{v1/2} dy$$

Or E(X) = $\frac{v2}{v2-2}$, v2>2

Observe that mean is independent of v1

3. **Other moments of the distribution**

On similar lines of simplifying E(X) we can show that

$E(X^2) = \dfrac{(v1+2)v_2^{\,2}}{v1(v2-2)(v2-4)}$

And

$V(X) = E(X^2)-[E(X)]^2 = \dfrac{2v_2^{\,2}(v1+v2-2)}{v1(v2-2)^2(v2-4)}$, v2>4

Also observe on similar lines

$(EX^K) = \left(\dfrac{v2}{v1}\right)^k \dfrac{\Gamma(v1/2+k)\Gamma(v2/2-k)}{\Gamma(v_1/2)\,\Gamma(v_2/2)}$

4. **Mode of the distribution**: The mode or modal value is that value of X for which the probability density function f(x) defined by (1) is maximum.

Consider $f(x) = \dfrac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})} \cdot \dfrac{x^{\frac{v1}{2}-1}}{\left(1+\frac{v1}{v2}x\right)^{(v1+v2)/2}}$

Or log f(x) = K+ (v1/2-1)logx –(v1+v2)/2 log(1+v1/v2x) where K= $\dfrac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})}$.

Now differentiation with respect to x we have

$$\frac{f`(x)}{f(x)} = \frac{\frac{v1}{2}-1}{x} - \frac{(v1+v2)}{2}\frac{v1}{v2}\frac{1}{(1+\frac{v1}{v2}x)}$$

Or

$$f`(x) = \left(\frac{\frac{v1}{2}-1}{x} - \frac{(v1+v2)}{2}\frac{v1}{v2}\frac{1}{(1+\frac{v1}{v2}x)}\right)f(x)$$

equating f`(x) = o implies

$$\left(\frac{\frac{v1}{2}-1}{x}-\frac{(v1+v2)}{2}\frac{v1}{v2}\frac{1}{(1+\frac{v1}{v2}x)}\right)=0$$

Or

$$\left(\frac{\frac{v1}{2}-1}{x}=\frac{(v1+v2)}{2}\frac{v1}{v2}\frac{1}{(1+\frac{v1}{v2}x)}\right)$$ **Solving we get**

X= $\frac{v2}{v1}\frac{v1-2}{v2+2}$ for v1>2 It can be easily verified that f``(x) <0. Therefore mode of the distribution is x= $\frac{v2}{v1}\frac{v1-2}{v2+2}$ for v1>2

Observe that the distribution is unimodal

Also observe that mode = $\frac{v2}{v2+2}\frac{v1-2}{v1}$ Hence for F distribution, mode is always less than unity

5. **Special cases.**
   The general probability density function of the *F* distribution is a bit complicated, but it simplifies in a couple of special cases.

1. If *v1*=2,

   $f(x)=1/(1+2x/v_2)^{1+v_2/2}$, $x\in(0,\infty)$

2. If *v1*=*v2*,

   $f(x)=\Gamma(v1)/\Gamma^2(v1/2).\ x^{n/2-1/}/(1+x)^n$, $x\in(0,\infty)$

3. If $v1=v2=2$,

   $f(x)=1/(1+x)^2$, $x\in(0,\infty)$

4. If $v1=v2=1$,

   $f(x)=1/\pi\sqrt{x}\,(1+x)$, $x\in(0,\infty)$

6. Some results

Result 1: Suppose that *X* has the *F* distribution with *v1* degrees of freedom in the numerator and *v2* degrees of freedom in the denominator. Then 1/*X* has the *F* distribution with *v2* degrees of freedom in the numerator and *v1* degrees of freedom in the denomination

Proof: The pdf of F with v1 and v2 degree of freedom is

$$f(x)=\frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})}\cdot\frac{x^{\frac{v1}{2}-1}}{\left(1+\frac{v1}{v2}x\right)^{(v1+v2)/2}},\quad 0\le x<\infty$$.

Making the substitution, y=1/x, hence dy=1/x^{2}dx

The pdf can be written using the transformation technique as

$$f(y)=\frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})}\cdot\frac{(1/y)^{\frac{v1}{2}-1}}{\left(1+\frac{v1}{v2}(1/y)\right)^{(v1+v2)/2}}\left(\frac{1}{y}\right)^2$$ **or**

$$f(y)=\frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})}\cdot\left(\frac{v2}{v1}\right)^{(v1+v2)/2}\frac{y^{(v1+v2)/2-\frac{v1}{2}+1-2}}{\left(1+\frac{v2}{v1}y\right)^{(v1+v2)/2}}$$

$$f(y)=\frac{\left(\frac{v2}{v1}\right)^{v2/2}}{B(\frac{v1}{2},\frac{v2}{2})}\cdot\frac{y^{v2/2-1}}{\left(1+\frac{v2}{v1}y\right)^{(v1+v2)/2}}$$, y>0

Which is pdf of F variate with v2 and v1 degree of freedom.

Or we can deduce that P(x(v1,v2)≥c)= P(x(v2,v1)≤1/c)

Result2. : Suppose that *T* has the *t* distribution with *n* degrees of freedom. Then $X=T^2$ has the *F* distribution with 1 and *n* degrees of freedom.**(Relation between T and F distribution)**

Proof: Let *T* has the *t* distribution with *n* degrees of freedom. The pdf of T is given by

$$f(t) = \frac{1}{B(\frac{1}{2},\frac{n}{2})\sqrt{n}} \frac{1}{\left(1+t^2/n\right)^{\frac{n+1}{2}}}$$

Consider the transformation $X=T^2$,
the inverse transformation in the interval $0<t<\infty$ is
$T=\sqrt{X}$, $dt/dx = 1/2\sqrt{x}$
The pdf of X using transformation technique is

$$\mathbf{g(x)} = \frac{1}{B(\frac{1}{2},\frac{n}{2})\sqrt{n}} \frac{1}{\left(1+x/n\right)^{\frac{n+1}{2}}} \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2} \frac{(\frac{1}{n})^{1/2}}{B(\frac{1}{2},\frac{n}{2})} \frac{x^{-1/2}}{\left(1+x/n\right)^{\frac{n+1}{2}}}$$

similarly in the range $-\infty<t<0$ the pdf of x is

$$\mathbf{f(x)} = \frac{1}{2} \frac{(\frac{1}{n})^{1/2}}{B(\frac{1}{2},\frac{n}{2})} \frac{x^{-1/2}}{\left(1+x/n\right)^{\frac{n+1}{2}}}$$

Combining the pdf of X is

f(x) = $\dfrac{\dfrac{(\frac{1}{n})^{1/2}}{B(\frac{1}{2},\frac{n}{2})}\, x^{-1/2}}{\left(1+x/n\right)^{\frac{n+1}{2}}}$ which is pdf of F variate with 1 and n degree of freedom.

Result3. : Suppose that *T* has the *t* distribution with *n* degrees of freedom. Then $X=1/T^2$ has the *F* distribution with n and *1* degrees of freedom.(Prove on similar lines)

Relation between exponential and F distribution:

Result 4. : Suppose that *X* and *Y* are independent random variables, each with the exponential distribution with rate parameter *λ*. Then *Z*=*X*/*Y*. has the *F* distribution with 2 and 2 degrees of .

Proof:

The joint pdf of X and Y is

$f(x,y) = \lambda^2 \; e^{-\lambda x} \; e^{-\lambda y}$

Consider the transformation

Z=X/Y, W=Y The inverse transformation is X=ZW, Y=W, The jacobian of the transformation is j= $\begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix}$ = w

The joint density of z and w is

$g(z,w) = \lambda^2 \; e^{-\lambda zw} e^{-\lambda w} \; w$

$= \lambda^2 \; e^{-\lambda w(1+z)} w, \;\; 0<z,w<0$

Now, integrating out with respect to w we get the distribution of z as

g(z) = $\displaystyle\int_0^\infty g(z,w)dw$

= $\lambda^2 \int_0^\infty e^{-\lambda w(1+z)} w dw$ By making the substitution λw(1+z) =t we get

= $\lambda^2 \int_0^\infty e^{-t} \frac{t}{\lambda(1+z)} \frac{dt}{\lambda(1+z}$ = $\lambda^2 \frac{1}{\lambda^2(1+z)^2} \int_0^\infty te^{-t} dt$

= $\frac{1}{(1+z)^2}$, z>0 which is pdf of F distribution with 2 and 2 degree of freedom.

7. **Application:**
    1. Testing the homogeneity of variance
    2. In the analysis of variance technique
    3. To test the equality of several regression coefficients

8. Conclusion: In this session we have defined the **Snedecor's** F random variable and derived its density function . We have derived the mean and mode of the distribution . We have studied the properties of the properties of the distribution. We have stated some simple forms of the distribution when the degree of freedom in the numerator and denominator changes. We have proved some results which explain the relation between some stated distributions. We have discussed the relation between F and T distribution. We have discussed the Relation between F and Chi-square distribution We have listed some applications of F distribution. We have learnt how to read F table.