Frequently asked questions:

1. Define F random Variate.

Let U and V are two independent random variables having chi-square distribution with v1 and v2 degree of freedom, then X= $\frac{U/v_1}{V/v_2}$ is a F random variable having v1 and v2 degree of freedom

1. Write the pdf of F variate

The pdf of F variate with v1 and v2 degree of freedom is

$$f(x) = \frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})} \cdot \frac{x^{\frac{v1}{2}-1}}{\left(1+\frac{v1}{v2}x\right)^{(v1+v2)/2}}, \ \ 0 < x < \infty$$

2. **Derive the pdf of F variate with v1 and v2 degree of freedom**

   Let U and V are independent random variables having chi-square distribution with v1 and v2 degree of freedom.

Then the joint density of U and V is given by

g(u,v) = g(u) . g(v)

= $\frac{1}{2^{v1/2}\Gamma(v1/2)} u^{\frac{v1}{2}-1} e^{-u/2} \frac{1}{2^{v2/2}\Gamma(v2/2)} v^{\frac{v2}{2}-1} e^{-v/2}$,u>0, v>0

Or $\frac{1}{2^{(v1+v2)/2}\Gamma(v1/2)} \frac{1}{\Gamma(v2/2)} u^{\frac{v1}{2}-1} v^{\frac{v2}{2}-1} e^{-(u+v)/2} \ \ u > 0, v > 0$

Consider the transformation

X= $\frac{U/v_1}{V/v_2}$ and Y=V

The inverse transformation is

U= $XY\frac{v_1}{v_2}$ V=Y

The jacobian of the transformation is

J = $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ = $\begin{vmatrix} \frac{yv1}{v2} & \frac{xv1}{v2} \\ 0 & 1 \end{vmatrix}$ = y v_1/v_2

Hence the joint density of X and Y is

f(x,y) =

$$\frac{1}{2^{(v1+v2)/2}\Gamma(v1/2)}\frac{1}{\Gamma(v2/2)}(xy\frac{v1}{v2})^{\frac{v1}{2}-1}y^{\frac{v2}{2}-1}e^{-(xy\frac{v1}{v2}+y)/2}y\frac{v_1}{v_2}$$

= $\dfrac{(\frac{v1}{v2})^{v1/2}}{2^{(v1+v2)/2}\Gamma(v1/2)}\dfrac{1}{\Gamma(v2/2)}(x)^{\frac{v1}{2}-1}y^{\frac{v1}{2}+\frac{v2}{2}-1}e^{-y/2(1+x\frac{v1}{v2})}$ for x>0 and y>0

Now, integrating out with respect to y we get the distribution of X

I.e. f(x) = $\int_0^\infty f(x,y)dy$

= $\dfrac{(\frac{v1}{v2})^{v1/2}}{2^{(v1+v2)/2}\Gamma(v1/2)}\dfrac{1}{\Gamma(v2/2)}(x)^{\frac{v1}{2}-1}\int_0^\infty y^{\frac{v1}{2}+\frac{v2}{2}-1}e^{-y/2(1+x\frac{v1}{v2})}\ dy$

By making the substitution w=($y/2(1+x\frac{v1}{v2}$) we get f(x) =

$$\frac{(\frac{v1}{v2})^{v1/2}}{2^{(v1+v2)/2}\Gamma(v1/2)}\ \frac{1}{\Gamma(v2/2)}(x)^{\frac{v1}{2}-1}\int_0^\infty\left(\frac{2w}{1+x\frac{v1}{v2}}\right)^{(\frac{v1+v2}{2}-1)} e^{-w}\frac{2}{1+x\frac{v1}{v2}}dw$$

$$= \frac{(\frac{v1}{v2})^{v1/2}}{2^{(v1+v2)/2}\Gamma(v1/2)}\ \frac{1}{\Gamma(v2/2)}(x)^{\frac{v1}{2}-1}\left(\frac{2}{1+x\frac{v1}{v2}}\right)^{(\frac{v1+v2}{2})}\int_0^\infty w^{(\frac{v1+v2}{2}-1)}e^{-w}dw$$

$$= \frac{(\frac{v1}{v2})^{v1/2}}{2^{(v1+v2)/2}\Gamma(v1/2)}\ \frac{1}{\Gamma(v2/2)}(x)^{\frac{v1}{2}-1}\left(\frac{2}{1+x\frac{v1}{v2}}\right)^{(\frac{v1+v2}{2})}\Gamma\left(\frac{v1+v2}{2}\right)$$

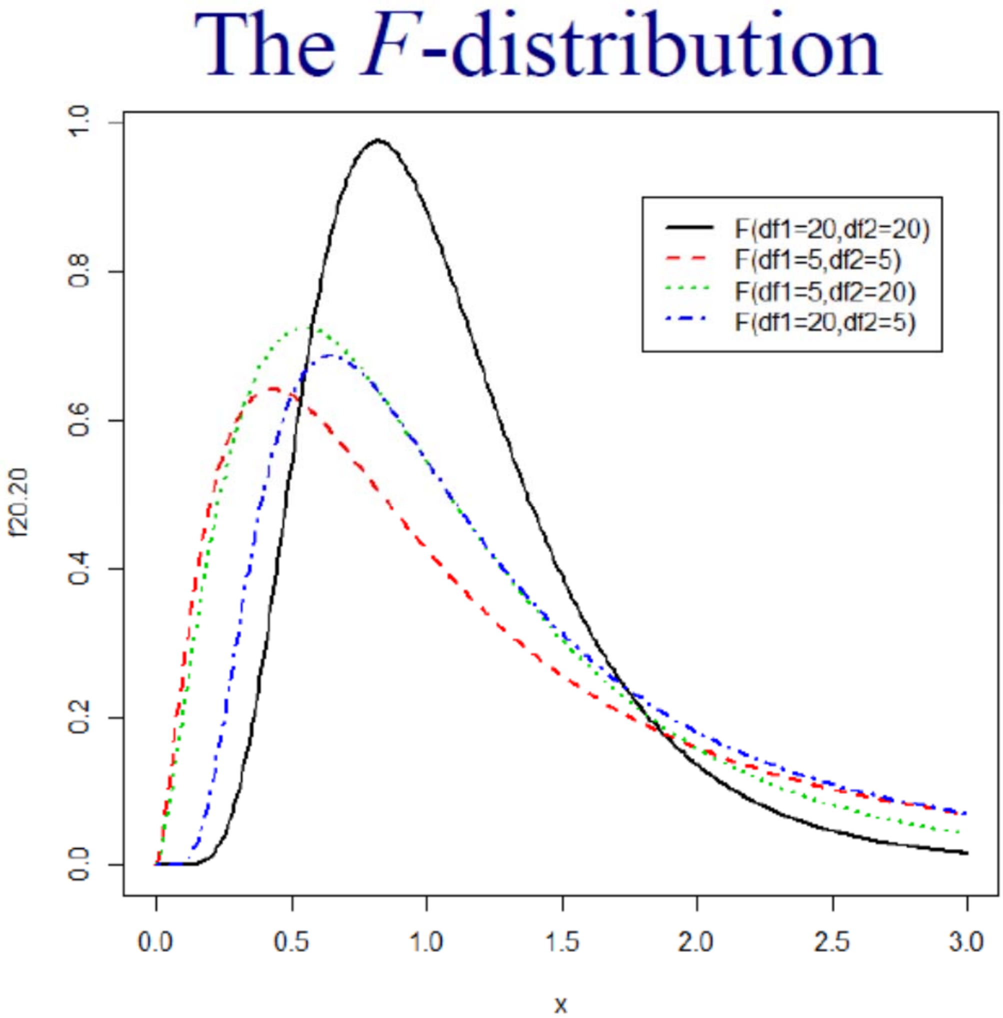
Or

$$f(x)=\frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})}\cdot\frac{x^{\frac{v1}{2}-1}}{\left(1+\frac{v1}{v2}x\right)^{(v1+v2)/2}},\ \ 0\le x<\infty$$

Using symbol we can write as X~F(v1, v2)

3. **Sketch the F-distribution probability curve :**

F-distributions are generally skewed. The shape of an *F*-distribution depends on the values of *v1* and *v2* the degrees of freedoms



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5. Derive the Mean of the distribution

Mean = E(X) = $\int_0^\infty x \; f(x)dx$

= $$\int_0^\infty x \frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})} \cdot \frac{x^{\frac{v1}{2}-1}}{\left(1+\frac{v1}{v2}x\right)^{(v1+v2)/2}} dx$$

= $$\frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})} \int_0^\infty x . \frac{x^{\frac{v1}{2}-1}}{\left(1+\frac{v1}{v2}x\right)^{(v1+v2)/2}} dx$$

By making the substitution 1/y=($1+\frac{v1}{v2}x$) we get or x= $\frac{1-y}{y}\frac{v2}{v1}$

E(X) = $\frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})}\int_0^1 \cdot\frac{(\frac{1-y}{y}\frac{v2}{v1})^{\frac{v1}{2}}}{(1/y)^{(v1+v2)/2}}(1/y^2)\frac{v2}{v1}dy$

= $\frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})}(\frac{v2}{v1})^{(v1/2+1)}\int_0^1 y^{\,v2/2-2}.(1-y)^{v1/2}dy$

= $\frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})}(\frac{v2}{v1})^{(v1/2+1)}B(\frac{v1}{2}+1,\frac{v2}{2}-1)$ **as** $B(\frac{v1}{2}+1,\frac{v2}{2}-1)$ = $\int_0^1 y^{\,v2/2-2}.(1-y)^{v1/2}dy$

Or E(X) = $\frac{v2}{v2-2}$, v2>2

Observe that mean is independent of v1

6. **Derive the Mode F distribution**:
   The mode or modal value is that value of X for which the probability density function f(x) is maximum.

Consider $f(x)=\frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})}\cdot\frac{x^{\frac{v1}{2}-1}}{\left(1+\frac{v1}{v2}x\right)^{(v1+v2)/2}}$

Or log f(x) = K+ (v1/2-1)logx –(v1+v2)/2 log(1+v1/v2x) where K= $\frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})}$.

Now differentiation with respect to x we have

$$\frac{f`(x)}{f(x)}=\frac{\frac{v1}{2}-1}{x}-\frac{(v1+v2)}{2}\frac{v1}{v2}\frac{1}{(1+\frac{v1}{v2}x)}$$

Or

$$f`(x)=(\frac{\frac{v1}{2}-1}{x}-\frac{(v1+v2)}{2}\frac{v1}{v2}\frac{1}{(1+\frac{v1}{v2}x)})f(x)$$

equating f`(x) = o implies

$$(\frac{\frac{v1}{2}-1}{x}-\frac{(v1+v2)}{2}\frac{v1}{v2}\frac{1}{(1+\frac{v1}{v2}x)})=0$$

Or

$$(\frac{\frac{v1}{2}-1}{x}=\frac{(v1+v2)}{2}\frac{v1}{v2}\frac{1}{(1+\frac{v1}{v2}x)})$$ **Solving we get**

X= $\frac{v2}{v1}\frac{v1-2}{v2+2}$ for v1>2 It can be easily verified that f``(x) <0. Therefore mode of the distribution is x= $\frac{v2}{v1}\frac{v1-2}{v2+2}$ for v1>2

Observe that the distribution is unimodal

Also observe that mode = $\frac{v2}{v2+2}\frac{v1-2}{v1}$ Hence for F distribution, mode is always less than unity

7. **Write the pdf of F distribution in the following cases**
   (i) 2 and n degree of freedom
   (ii) N and n degree of freedom
   (iii) 2 and 2 degree of freedom
   (iv) 1 and 1 degree of freedom

(i) If $v1=2$,

$f(x)=1/(1+2x/n)^{1+n/2}$, $x\in(0,\infty)$

(ii) If $v1=v2$ say $=n$,

$f(x)=\Gamma(n)/\Gamma^2(n/2).\ x^{n/2-1/}/(1+x)^n$, $x\in(0,\infty)$

(iii) If $v1=v2=2$,

$f(x)=1/(1+x)^2$, $x\in(0,\infty)$

(iv) If $v1=v2=1$,

$f(x)=1/\pi\sqrt{x}\ (1+x)$, $x\in(0,\infty)$

8. : **Suppose that *X* has the *F* distribution with *v1* degrees of freedom in the numerator and *v2* degrees of freedom in the denominator. Then prove that 1/*X* has the *F* distribution with *v2* degrees of freedom in the numerator and *v1* degrees of freedom in the denomination**

The pdf of F with v1 and v2 degree of freedom is

$$f(x)=\frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})}\cdot\frac{x^{\frac{v1}{2}-1}}{\left(1+\frac{v1}{v2}x\right)^{(v1+v2)/2}},\ 0\le x<\infty$$

.

Making the substitution, y=1/x, hence $dy=1/x^2dx$

The pdf can be written using the transformation technique as

$$f(y)=\frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})}\cdot\frac{(1/y)^{\frac{v1}{2}-1}}{\left(1+\frac{v1}{v2}(1/y)\right)^{(v1+v2)/2}}(\frac{1}{y})^2$$ **or**

$$f(y)=\frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})}.(\frac{v2}{v1})^{(v1+v2)/2}\frac{y^{(v1+v2)/2-\frac{v1}{2}+\ 1-2}}{\left(1+\frac{v2}{v1}y\right)^{(v1+v2)/2}}$$

$$f(y)=\frac{\left(\frac{v2}{v1}\right)^{v2/2}}{B(\frac{v1}{2},\frac{v2}{2})}.\frac{y^{v2/2-1}}{\left(1+\frac{v2}{v1}y\right)^{(v1+v2)/2}}$$, y>0

Which is pdf of F variate with v2 and v1 degree of freedom**.**

9. . : **Suppose that *T* has the [*t* distribution] with *n* degrees of freedom. Then prove that *X*=T^2 has the *F* distribution with 1 and *n* degrees of freedom.**

   Let *T* has the [*t* distribution] with *n* degrees of freedom. The pdf of T is given by

f(t) = $$\frac{1}{B(\frac{1}{2},\frac{n}{2})\sqrt{n}}\frac{1}{\left(1+t^2/n\right)^{\frac{n+1}{2}}}$$

Consider the transformation X=T^2,
the inverse transformation in the interval 0<t<∞is
T=√X, dt/dx = 1/2√x
The pdf of X using transformation technique is

g(x) = $$\frac{1}{B(\frac{1}{2},\frac{n}{2})\sqrt{n}}\frac{1}{\left(1+x/n\right)^{\frac{n+1}{2}}}\ \frac{1}{2\sqrt{x}}$$

= $$\frac{1}{2}\frac{(\frac{1}{n})^{1/2}}{B(\frac{1}{2},\frac{n}{2})}\frac{x^{-1/2}}{\left(1+x/n\right)^{\frac{n+1}{2}}}$$

similarly in the range -∞<t<0 the pdf of x is

$$\frac{1}{2}\frac{(\frac{1}{n})^{1/2}}{B(\frac{1}{2},\frac{n}{2})}\frac{x^{-1/2}}{\left(1+\frac{x}{n}\right)^{\frac{n+1}{2}}}$$

Combining the pdf of X is

f(x) = $\frac{(\frac{1}{n})^{1/2}}{B(\frac{1}{2},\frac{n}{2})}\frac{x^{-1/2}}{\left(1+\frac{x}{n}\right)^{\frac{n+1}{2}}}$ which is pdf of F variate with 1 and n degree of freedom.

10. : **Suppose that *X* and *Y* are independent random variables, each with the exponential distribution with rate parameter *λ*. Then prove that *Z=X/Y*. has the *F* distribution with 2 and 2 degrees of freedom.**

The joint pdf of X and Y is

$f(x,y) = \lambda^2 \ e^{-\lambda x} \ e^{-\lambda y}$

Consider the transformation

Z=X/Y, W=Y The inverse transformation is X=ZW, Y=W, The jacobian of the transformation is j= $\begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix}$ = w

The joint density of z and w is

$g(z,w) = \lambda^2 \ e^{-\lambda zw} e^{-\lambda w} \ w$

$= \lambda^2 \ e^{-\lambda w(1+z)} w, \ 0<z,w<0$

Now, integrating out with respect to w we get the distribution of z as

g(z) = $\int_0^\infty g(z,w)dw$

$= \lambda^2 \int_0^\infty e^{-\lambda w(1+z)} w\,dw$ By making the substitution λw(1+z) =t we get

$= \lambda^2 \int_0^\infty e^{-t} \frac{t}{\lambda(1+z)} \frac{dt}{\lambda(1+z} = \lambda^2 \frac{1}{\lambda^2(1+z)^2} \int_0^\infty te^{-t}dt$

$= \frac{1}{(1+z)^2}$, z>0 which is pdf of F distribution with 2 and 2 degree of freedom.

11. If X follow F distribution with v1 and v2 degree of freedom and if we let v2 -> ∞ then Y= v1X follow chi-square distribution with v1 degree of freedom

The pdf of X is

$$f(x) = \frac{\left(\frac{v1}{v2}\right)^{v1/2}}{B(\frac{v1}{2},\frac{v2}{2})} \cdot \frac{x^{\frac{v1}{2}-1}}{\left(1+\frac{v1}{v2}x\right)^{(v1+v2)/2}},\ 0 \le x < \infty \quad (*)$$

Taking limit as v2 -> ∞ we have $\frac{\Gamma n + k}{n} \longrightarrow n^k$

Hence $\frac{\Gamma(v1+v2)/2}{v2^{v1/2}\Gamma(v2/2)} \longrightarrow \frac{(v2/2)^{v1/2}}{v2^{v1/2}} = \frac{1}{2^{v1/2}}$ (1)

Also $\underset{v2 \longrightarrow \infty}{lt} (1+\frac{v1}{v2}x)^{(v1+v2)/2}$

=

$$\underset{v2 \longrightarrow \infty}{lt} [(1+\frac{v1}{v2}x)^{v2}]^{v1/2} \quad \underset{v2 \longrightarrow \infty}{lt} (1+\frac{v1}{v2}x)^{v1/2}$$

=exp(v1x/2) = exp(y/2) …(2) as taking the transformation v1x=y

Using 1 and 2 in * we get the pdf of Y as g(y) = $\frac{(v1/2)^{v1/2}}{\Gamma(v1/2)}(y/v1)^{(v1/2-1)}\frac{1}{v1}\exp(-y/2)$

$$\frac{}{2^{v1/2}\Gamma(v1/2)}(y)^{(v1/2-1)}\exp(-y/2), 0 < y < \infty$$

Which is the pdf of chi square variate with v1 degree of freedom.

12. Write the Application of F distribution:

The applications are

1. Testing the homogeneity of variance
2. In the analysis of variance technique
3. To test the equality of several regression coefficients