Frequently Asked Questions

1. **Define Student's t – variate**

   Answer : Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from $N(\mu, \sigma^2)$ population. Then the student's t-statistic is defined as

   $t = \dfrac{\bar{x} - \mu}{S/\sqrt{n}}$ ~Student's t distribution with (n-1) d.f. with p.d.f. defined by

   $$f(t) = \frac{1}{\sqrt{n-1}\,B(\frac{1}{2}, \frac{n-1}{2})} \frac{1}{\left(1+\frac{t^2}{n-1}\right)^{n/2}}; \; for \; -\infty < t < \infty.$$

2. **Define Fisher's t-variate** : According to Fisher, the statistic 't' is the ratio of a standard normal variate to the square root of an independent chi-square variate divided by its degree of freedom. Thus, the statistic 't' is defined as

   $t = \dfrac{Z}{\sqrt{X/n}}$ ~student's t - distribution with n d.f. where Z ~N(0, 1) and X~ $\chi^2_{(n)}$ d.f. such that the pdf is

   $$f(t) = \frac{1}{\sqrt{n}\,B(\frac{1}{2}, \frac{n}{2})} \cdot \frac{1}{\left(1+\frac{t^2}{n}\right)^{(n+1)/2}}, \quad -\infty < t < \infty$$

3. **Write the properties of t- distribution.**

   Answer : Let $t \sim t_{(n)}$ d.f. then

   a. When n = 1, *f(t)* reduces to

   $$f(t) = \frac{1}{\pi} \frac{1}{\left(1+t^2\right)}; \quad -\infty < t < \infty,$$

   which is the pdf of Cauchy distribution, i.e., t-- > Cauchy distribution, when n =1.

   b. Mean =0 and variance= n/(n-2), for n > 2.

   c. Mean deviation about mean is

$$\frac{\sqrt{n}\,\Gamma[(n-1)/2]}{\sqrt{\pi}\,\Gamma(n/2)}$$

d. All odd ordered moments are zero and all the even ordered moments exist and are constants.

4. **Derive the p.d.f. of Student's t- variate.**

Answer : Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from N(μ, σ^2) population. Then the student's t-statistic is defined as

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} \sim \text{Student's t distribution with (n-1) d.f.} \qquad (1)$$

Equation (1) can be written as

$$t^2 = \frac{n(\bar{x} - \mu)^2}{S^2} = \frac{n(\bar{x} - \mu)^2}{ns^2/(n-1)}$$

$$\Rightarrow \frac{t^2}{n-1} = \frac{(\bar{x} - \mu)^2}{\sigma^2/n} \times \frac{1}{ns^2/\sigma^2} = \frac{(\bar{x} - \mu)^2/(\sigma^2/n)}{ns^2/\sigma^2}.$$

Since $x_1, x_2, \ldots, x_n$ be a random sample of size n from N(μ, σ^2) population, then $\bar{x} \sim$ N(μ, σ^2/n). $\Rightarrow \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim$ N(0, 1). Hence $\frac{(\bar{x} - \mu)^2}{\sigma^2/n}$ being the square of a standard normal variate, is a chi-square variate with 1 d.f. Also, (ns^2/σ^2) is a chi-square variate with (n-1) d.f. Further, since $\bar{x}$ and s^2 are independently distributed , $\frac{t^2}{n-1}$ being the ratio of two independent χ^2 variates with 1 and (n - 1) d.f. respectively, is a β_2[1/2, (n-1)/2] variate and its distribution is given by

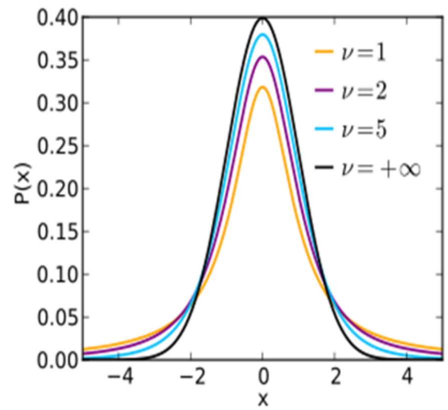
$$dF(t) = \frac{1}{B(\frac{1}{2}, \frac{n-1}{2})} \cdot \frac{\left(\frac{t^2}{n-1}\right)^{1/2 \;\; 1}}{\left(1 + \frac{t^2}{n-1}\right)^{n/2}} dt; \qquad 0 < t^2 < \infty$$

$$= \frac{1}{\sqrt{n-1}\,B(\frac{1}{2}, \frac{n-1}{2})} \frac{1}{\left(1 + \frac{t^2}{n-1}\right)^{n/2}} dt; \qquad -\infty < t < \infty.$$

Which is the p.d.f. of Student's t-distribution with (n-1) d.f. where the factor 2 disappears, since the integral form -∞ < t < ∞ must be unity.

5. **Write a note on probability curve of t - distribution.**
   Answer :

The *t*-distribution is symmetric and bell-shaped, like the normal distribution, but has heavier tails, that is, it is more flat to produce values that fall, far from its mean. That is f(-t)=f(t), the probability curve is symmetrical about the line t=0. As t increases, f(t) decreases rapidly. Here we observe that, for the degrees of freedom n = 1 and 2, the probability curve show more flat than when n-- >∞.



6. **Give some of the applications of t- distribution**.

Answer : The t- distribution has a wide range of applications, some of which are

i. To test the significance of sample mean from the population mean or simply, t-test for single mean.
ii. To test the significance of the difference between two sample means.
iii. To test the significance of an observed sample correlation coefficient and the sample regression coefficient.
iv. To test the significance of an observed partial correlation coefficient.

7. **Derive the rth moments of t-distribution and hence find its mean and variance**

Answer : Since f(t) is symmetric about the line t=0, all the moments of odd order about origin vanish, that is

$\mu'_{2r+1}(about\ origin) = 0,\ r = 0,1,2,3,...$

In particular, $\mu'_1(about\ origin) = 0 = mean.$ therefore the central moments coincide with moment about origin => $\mu_{2r+1} = 0,\ for\ r = 0,1,2,3,...$

The moments of even order are given by

$$\mu_{2r} = \mu'_{2r}(about\,origin) = \int_{-\infty}^{\infty} t^{2r} f(t)dt = 2\int_{0}^{\infty} t^{2r} f(t)dt$$

$$\therefore \mu_{2r} = 2.\frac{1}{\sqrt{n}\ B(\frac{1}{2},\frac{n}{2})} \int_0^\infty \frac{t^{2r}}{{1+\frac{t^2}{n}}^{(n+1)/2}}\, dt$$

The above integral is convergent if 2r < n.

Now let $1+\frac{t^2}{n} = \frac{1}{x} \Rightarrow t^{\ \ 2} = \frac{n(1\ \ x)}{x} \Rightarrow 2tdt = \ \ \frac{n}{x^2}\, dx$

When t = 0, y = 1 and when t = ∞, y = 0. Therefore, we have

$$\therefore \mu_{2r} = 2.\frac{1}{\sqrt{n}\ B(\frac{1}{2},\frac{n}{2})} \int_1^0 \frac{t^{2r}}{\frac{1}{x}^{(n+1)/2}} . \frac{n}{2tx^2}\, dx$$

$$= \frac{\sqrt{n}}{B(\frac{1}{2},\frac{n}{2})} \int_0^1 (t^2)^{(2r\ \ 1)/2} . x^{\frac{(n+1)}{2}\ \ 2}\ \ dx$$

$$= \frac{\sqrt{n}}{B(\frac{1}{2},\frac{n}{2})} \int_0^1 \frac{n(1\ \ x)}{x}^{r\ \ 1/2} . x^{\frac{(n+1)}{2}\ \ 2}\ \ dx$$

$$= \frac{n^r}{B(\frac{1}{2},\frac{n}{2})} \int_0^1 (1\ \ x)^{r-1/2} . x^{\frac{n}{2}-r-1}\ \ dx$$

$$= \frac{n^r}{B(\frac{1}{2},\frac{n}{2})}\, B(\tfrac{n}{2}\ \ r,\ r+\tfrac{1}{2})$$

$$\mu_{2r} = \frac{n^r \Gamma[(n/2)\ \ r].\Gamma(r+1/2)}{\Gamma(\frac{1}{2}).\Gamma(\frac{n}{2})}$$

$$= \frac{n^r(r\ \ \frac{1}{2})(r\ \ \frac{3}{2}).\dots\frac{3}{2}.\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(\frac{n}{2}\ \ r)}{\Gamma(\frac{1}{2}).(\frac{n}{2}\ \ 1).(\frac{n}{2}\ \ 2)\dots(\frac{n}{2}\ \ r)\Gamma(\frac{n}{2}\ \ r)}$$

$$= \frac{n^r(2r\ \ 1)(2r-3).\dots3.1}{(n\ \ 2).(n\ \ 4)\dots(n\ \ 2r)},\ n > 2r.$$

In particular,

$$\mu_2 = n\frac{1}{n\quad 2} = \frac{n}{n\quad 2} = \text{var}\,iance,\ \ for\ \ n > 2.$$

8. **Derive the expression for mean deviation about mean of t-distribution:**

Solution : Since t ~$t_{(n)}$ d.f. E(t) = 0 and hence,

Mean deviation (about mean)= $\int_{\infty}^{\infty} |t|\, f(t)dt$

$$= 2.\frac{1}{\sqrt{n}\, B(\frac{1}{2},\frac{n}{2})}\int_0^{\infty}\frac{t}{1+\frac{t^2}{n}^{\ (n+1)/2}}\,dt$$

$$= \frac{\sqrt{n}}{B(\frac{1}{2},\frac{n}{2})}\int_0^{\infty}\frac{1}{(1+x)^{(n+1)/2}}\,dx,\ \ where\ \ x = \frac{t^2}{n}$$

$$= \frac{\sqrt{n}}{B(\frac{1}{2},\frac{n}{2})}\int_0^{\infty}\frac{x^{1\ \ 1}}{(1+x)^{(n\quad 1)/2\,+\,1}}\,dx,$$

$$= \frac{\sqrt{n}}{B(\frac{1}{2},\frac{n}{2})}.B(\tfrac{n\ \ 1}{2},1) = \frac{\sqrt{n}\,\Gamma[(n\quad 1)/2]}{\sqrt{\pi}\ \ \Gamma(/\,2)}.$$

9. **Derive the limiting form of t- distribution**

Proof: Given t ~ $t_{(n)}$ d.f. then we have

$$f(t) = \frac{1}{\sqrt{n}\, B(\frac{1}{2},\frac{n}{2})}\,\frac{1}{1+\frac{t^2}{n}^{\ (n+1)/2}},\qquad -\infty < t < \infty$$

$$\therefore \lim_{n\to\infty} f(t) = \lim_{n\to\infty}\left[\frac{1}{\sqrt{n}\, B(\frac{1}{2},\frac{n}{2})}\frac{1}{\left(1+\frac{t^2}{n}\right)^{(n+1)/2}}\right]$$

$$= \lim_{n\to\infty}\frac{\Gamma[(n+1)/2]}{\sqrt{n}\,(\Gamma\frac{1}{2}\times\Gamma\frac{n}{2})}.\lim_{n\to\infty}\left[\frac{1}{\left(1+\frac{t^2}{n}\right)^{(n+1)/2}}\right]$$

Since, $\Gamma\frac{1}{2}=\sqrt{\pi}, \quad and \ \lim_{n\to\infty}\frac{\Gamma(n+k)}{\Gamma n}=n^k$ and therefore we have

$$\lim_{n\to\infty} f(t) = \frac{1}{\sqrt{2\pi}}.\lim_{n\to\infty}\left[\left(1+\frac{t^2}{n}\right)^{n}\right]^{1/2}\times\lim_{n\to\infty}\left(1+\frac{t^2}{n}\right)^{1/2}$$

$$\lim_{n\to\infty} f(t) = \frac{1}{\sqrt{2\pi}}.e^{\frac{t^2}{2}}, \quad \infty< t < \infty.$$

which is the pdf of a standard normal variate. Hence, for large n, that is, when t-- > ∞, t-> N(0, 1) distribution asymptotically.

10. Write a note on critical values of t-distribution

Answer : The critical or significant values of t at level of significance α and d.f.(υ = n-1) for two tailed test are given by the equation

$P[\ |t| > t_\upsilon(\alpha)\ ] = \alpha \quad \Rightarrow P[\ |t| \le t_\upsilon(\alpha)\ ] = 1-\alpha.$

The values of $t_\upsilon(\alpha)$ can be obtained from student's t-table. Since t- distribution is symmetric about t = 0, we have

$$P[\ t > t_\upsilon(\alpha)\ ] + P[\ t < -t_\upsilon(\alpha)] = \alpha$$

$\Rightarrow 2.\ P[\ t > t_\upsilon(\alpha)\ ] = \alpha \ \Rightarrow P[\ t > t_\upsilon(\alpha)\ ] = \alpha/2.$

Therefore, $P[\ t > t_\upsilon(2\alpha)\ ] = \alpha$,

where, $t_\upsilon(2\alpha)$ gives the significant value of t for a singe tail test(right or left) at level of significance α and d.f.(υ).