Subject	Statistics
Year	1st Year B.Sc
Paper no	05
Paper Name	Probability Distributions-1
Topic no	19
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E-Learning Module on Covariance

Learning Objectives

By the end of this session, you will be able to:

- Explain the meaning and definition of covariance
- Explain the properties of covariance
- Explain the interpretation of value obtained in covariance
- Explain the uncorrelatedness and independence
- Explain the limitations of covariance

Meaning

Covariance is a measure of how much two random variables change together. i.e. Covariance provides a measure of the strength of the correlation between two or more sets of random variates.

If the greater values of one variable mainly correspond with the greater values of the other variable and the same holds for the smaller values, i.e. the variables tend to show similar behavior, the covariance is positive.

In the opposite case, when the greater values of one variable mainly correspond to the smaller values of the other, i.e. the variables tend to show opposite behavior, the covariance is negative.

The sign of the covariance shows the tendency in the linear relationship between the variables. The magnitude of the covariance is not that easy to interpret. However, the normalized version of the covariance, the correlation coefficient, shows by its magnitude the strength of the linear relation.

A distinction must be made between

- The covariance of two random variables, which is a population parameter that can be seen as a property of the joint probability distribution.

- The sample covariance, which serves as an estimated value of the parameter.

Definition

Covariance of two random variables X and Y, each with sample size n is defined by the expectation value. i.e. If X and Y are two random variables, then covariance between them is defined as,

$$
\begin{aligned}
Cov(X,Y) &= E[\{(X-E(X)\}\{Y-E(Y)\}] \\
&= E[XY-YE(X)-XE(Y)+E(X)E(Y)] \\
&= E(XY)-E(Y)E(X)-E(X)E(Y)+E(X)E(Y) \\
&= E(XY)-E(X)E(Y)
\end{aligned}
$$

It can be also denoted by σ_{xy}.

If X and Y are independent, then $E(XY)=E(X)E(Y)$.

Hence, in this case,
$Cov(X,Y)=E(X)E(Y)-E(X)E(Y) =0$

Properties

1. If X and Y are two random variables, a and b are constants then, Cov(aX, bY)=ab.Cov(X, Y).

Proof:

$$
\begin{aligned}
Cov(aX, bY) &= E[\{(aX-E(aX)\}\{bY-E(bY)\}] \\
&= E[\{(aX-aE(X)\}\{bY-bE(Y)\}] \\
&= abE[\{(X-E(X)\}\{Y-E(Y)\}] \\
&= ab.Cov(X, Y)
\end{aligned}
$$

2. If X and Y are two random variables, a and b are constants then, Cov(aX, bY)=Cov(X, Y)

Proof:

$$\begin{aligned}
Cov(X+a, Y+b) &= E[\{(X+a-E(X+a)\}\{Y+b-E(Y+b)\}] \\
&= E[\{(X+a-a-E(X)\}\{Y+b-b-E(Y)\}] \\
&= E[\{(X-E(X)\}\{Y-E(Y)\}] \\
&= Cov(X, Y)
\end{aligned}$$

3. If X and Y are two random variables and a, b, c and d are constants then,

Cov(aX+b, cY+d)=ac.Cov(X, Y)

Proof:

Cov(aX+b, cY+d)

=E[{(aX+b-E(aX+b)}{cY+d-E(cY+d)}]

=E[{(aX+b-a-bE(X)}{Y+c-c-dE(Y)}]

=ac.E[{(X-E(X)}{Y-E(Y)}]

=ac.Cov(X, Y)

4. Covariance of a random variable with itself i.e. if we consider Y=X then, Cov(X, X)=V(X).

Proof:

Cov(X, X)=E[{(X-E(X)}{X-E(X)}]

$=E[X-E(X)]^2=V(X)$

Hence, when we take both the variables same, we can consider Variance of the distribution as a particular case of Covariance.

5. Property of Symmetry states that $Cov(X,Y)=Cov(Y,X)$

Proof:

$$
\begin{aligned}
Cov(X,Y) &= E[\{(X-E(X)\}\{Y-E(Y)\}] \\
&= E[\{(Y-E(Y)\}\{X-E(X)\}] \\
&= Cov(Y, X)
\end{aligned}
$$

6. Property of bilinearity of Covariance:

Let X, Y and Z are real valued random variables and a, b and c are constants, then consider

$Cov(aX+bY, cZ)$

$=E[\{aX+bY-E(aX+bY)\}\{cZ-E(cZ)\}]$

$=E[\{aX-aE(X)+bY-bE(Y)\}\{cZ-cE(Z)\}]$

$=E[\{aX-aE(X)\}\{cZ-cE(Z)\}]$

$+E[\{bY-bE(Y)\}\{cZ-cE(Z)\}]$

$=acE[\{X-E(X)\}\{Z-E(Z)\}]+bcE[\{Y-E(Y)\}\{Z-E(Z)\}]$

$=acCov(X, Z)+bcCov(Y, Z)$ -------------------(1) is a linear function.

By the property of symmetry, $Cov(cZ, aX+bY)=acCov(Z, X)+bcCov(Z, Y)$------(2), is also a linear function.

From (1) and (2), we can say that covariances are bilinear.

Hence, in general, if we can take X_1, X_2, …X_K and Y be K+1 random variables and a_1, a_2, …, a_k and c are constants then,

$Cov(\Sigma a_i X_i, cY)=c\Sigma a_i Cov(X_i, Y)$

7. If X_1, X_2, …,X_n, Y_1, Y_2, …Y_m are random variables, then

$$Cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, Y_j)$$

Proof:

We know that $Cov(\Sigma a_i X_i, cY) = c\Sigma a_i Cov(X_i, Y)$

Suppose we take all a_i's and c to be 1,

$Cov(\Sigma X_i, Y) = \Sigma Cov(X_i, Y)$-------(1)

Now consider,

$$Cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j) = \sum_{i=1}^{n} Cov(X_i, \sum_{j=1}^{m} Y_j)$$

$$= \sum_{i=1}^{n} Cov(\sum_{j=1}^{m} Y_j, X_i)$$ (By property of symmetry)

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} Cov(Y_j, X_i)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} Cov(X_i, Y_j)$$

Hence the proof.

8. If X, Y, U and V are real valued random variables and a, b, c and d are constants, then $Cov(aX+bY, cU+dV)=ac\ Cov(X, U)+ ad\ Cov(X, V)+bc\ Cov(Y, U)+ bd\ Cov(Y, V)$

Proof: Consider,

$Cov(aX+bY, cU+dV)$

$=E[\{aX+bY-E(aX+bY)\}\{cU+dV-E(cU+dV)\}]$

$=E[\{a(X-E(X))+b(Y-E(Y)\}\{c(U-E(U))+d(VE(V))\}]$

$=E[ac\{X-E(X)\}\{U-E(U)\}]+E[ad\{X-E(X)\}\{V-E(V)\}]+E[bc\{Y-E(Y)\}\{U-E(U)\}]+E[bd\{Y-E(Y)\}\{V-E(V)\}]$

$=ac\ Cov(X, U)+ ad\ Cov(X, V)+bc\ Cov(Y, U)+ bd\ Cov(Y, V)$

Interpretation of Value of Covariance

Positive covariance – indicates that higher than average values of one variable tend to be paired with higher than average values of the other variable.

Negative covariance – indicates that higher than average values of one variable tend to be paired with lower than average values of the other variable.

Zero covariance – if the two random variables are independent, the covariance will be zero.

Uncorrelatedness and Independence

If *x* and *y* are independent, then their covariance is zero. This follows because under independence, E(XY)=E(X)E(Y)

However, the converse is not generally true. For example, let *x* be uniformly distributed in [-1, 1] and let $y=x^2$. Clearly, *X* and Y are dependent, but

$$\begin{aligned}Cov(X, Y) &= Cov(X, X^2)\\ &= E(X.X^2)-E(X)E(X^2)\\ &= E(X^3)- E(X)E(X^2)\\ &= 0-0.E(X^2)=0\end{aligned}$$

In this case, the relationship between y and x is non-linear, while correlation and covariance are measures of linear dependence between two variables. Still, as in the example, if two variables are uncorrelated, that does not imply that they are independent.

Limitations

Since the number representing covariance depends on the units of the data, it is difficult to compare covariances among data sets having different scales. A value that might represent a strong linear relationship for one data set might represent a very weak one in another. Hence, the correlation coefficient addresses this issue by normalizing the covariance to the product of the standard deviations of the variables, creating a dimensionless quantity, which facilitates the comparison of different data sets.

Illustration - 1

Let X and Y be 2 random variables such that $V(X)=2$ and $Cov(X, Y)=1$. Obtain the $Cov(5X, 2X+3Y)$

Solution: Let us use the property, $Cov(aX+bY, cZ)$ and $Cov(X, X)=V(X)$.

Consider

$$Cov(5X, 2X+3Y)=Cov(5X, 2X)+Cov(5X, 3Y)$$
$$=10V(X)+15Cov(X, Y)$$
$$=10.2+15.1=35$$

Illustration - 2

Two random variables X and Y have the following joint probability density function. $f(x,y)=(4-x-y)/8; 0 \le x \le 2,\ 0 \le y \le 2$. Find Cov(X, Y)

Solution:

Cov(X, Y)=E(XY)-E(X)E(Y)

Before we find Expectation and variance, we find marginal distribution of X and Y.

Marginal distribution of X is given by,

$$f(x) = \frac{1}{8}\int_0^2 (4-x-y)dy = \frac{1}{8}(6-2x) = \frac{1}{4}(3-x); 0 \le x \le 2$$

Marginal distribution of Y is given by,

$$f(y) = \frac{1}{8}\int_0^2 (4 - x - y)dx = \frac{1}{8}(6 - 2y) = \frac{1}{4}(3 - y);0 \le y \le 2$$

Now, let us obtain Expectation.

$$E(x) = \int_0^2 xf(x)dx = \frac{1}{4}\int_0^2 x(3 - x)dx = \frac{5}{6}$$

$$E(y) = \int_0^2 yf(y)dy = \frac{1}{4}\int_0^2 y(3 - y)dy = \frac{5}{6}$$

$$E(XY) = \int_0^2\int_0^2 xyf(,y)dxdy = \frac{1}{8}\int_0^2 y\left[\int_0^2 x(4 - x - y)dx\right]dy$$

$$= \frac{1}{8}\int_0^2 y\left(\frac{16}{3} - 2y\right)dy = \frac{2}{3}$$

Hence, Cov(X,Y)= $^2/_3 -(^5/_6)(^5/_6)=-^1/_{36}$