1. Introduction

Welcome to the series of E-learning modules on Cauchy and Logistic Distributions.

By the end of this session, you will be able to:

- Explain Standard Cauchy Distribution
- Explain Cauchy Distribution with two parameters – median, mode and moments
- Explain Standard Logistic Distribution – mgf, moments, skewness and kurtosis
- Explain Logistic Distribution with 2 parameters – mean and variance

2. Standard Cauchy Distribution

Definition

A random variable X is said to have a Standard Cauchy Distribution, if its probability density function is given by,

f of x is equal to 1 divided by pi into 1 plus x square, where x lies between minus infinity and infinity.

And X is termed as standard Cauchy variate.

More generally, Cauchy distribution with parameters lambda and mu has the probability density function,

f of y is equal to a lambda divided by pi into lambda square plus y minus mu the whole square, where y lies between minus infinity and infinity; and lambda greater than zero.

And we write X follows Cauchy (lambda, mu).

By substituting, X is equal to (Y minus μ) divide by lambda in the general probability density function of Cauchy distribution; we get the probability density function of standard Cauchy distribution.

Hence if Y follows, Cauchy (lambda, mu) then X is equal to (Y minus mu) divided by lambda follows Cauchy (zero, 1), i.e., standard Cauchy distribution.

3. Median, Mode and Moments

First, let us find the median of the distribution.
Let Y follows Cauchy distribution with parameters lambda and mu, and if M is the median of the distribution, then
Integral from minus infinity to M f of y dy is equal to half is equal to integral from M to infinity f of y dy
We can solve any of the integral. For simplicity let us consider the 2^{nd} integral, integral from M to infinity f of y dy is equal to half.
The range of the above integral can be split into two parts, namely, from M to mu and mu to infinity.

Hence, we get,
Integral from M to mu f of y dy plus integral from mu to infinity f of y dy is equal to half. Name it as equation 1.
Now, consider the 2^{nd} integral from 1.
That is, integral from mu to infinity f of y dy
Is equal to integral from mu to infinity lambda divided by pi into lambda square plus y minus mu the whole square dy
Now, substitute x is equal to y minus mu divided by lambda. As y is equal to infinity, x is also equal to infinity and as y is equal to mu, x is equal to zero.
Therefore, integral from mu to infinity, f of y dy is equal to 1 by pi into 1 divided by 1 plus x square dx is equal to half

Substituting this value in equation 1, we get
Integral from M to mu f of y dy plus half is equal to half
Implies integral from M to mu f of y dy is equal to zero, which is possible only when both upper and lower range of the integral is same.
Implies M is equal to mu.

Mode is the value for which f of (y) is maximum.
We can find the maximum by differentiating the probability density function with respect to y, then equating it zero and solving form y. Then, obtain the 2^{nd} derivative which should be negative.
If Y follows Cauchy lambda mu, then
f of y is equal to lambda divided by phi into lambda square plus y minus mu the whole square, where y lies between minus infinity to infinity and lambda greater than zero
To simplify the differentiating, let us work with logarithm.

By taking log on both the sides, we get,
Log f of (y) is equal to log lambda minus log phi minus log [lambda square minus (y minus mu) the whole square]
On differentiating we get,
f dash of (y) divided by f of (y) is equal to 2 into (y minus mu) divided by [lambda square minus (y minus mu) the whole square]
f dash of (y) is equal to zero implies y is equal to mu.
If we find 2^{nd} derivative at y is equal to mu, then it is negative.

Hence, y is equal to mu is the mode of the distribution.

Now, let us find the moments of the distribution.
If Y follows Cauchy (lambda, mu) then, mean of the distribution is given by,
Expectation of Y is equal to integral from minus infinity to infinity, y into f of y dy
Is equal to lambda by phi into integral from minus infinity to infinity, y divided by lambda square plus y minus mu the whole square dy
By adding and subtracting mu in the numerator, we get
lambda by phi into integral from minus infinity to infinity, y minus mu, plus mu whole divided by lambda square plus y minus mu the whole square dy.
by splitting the integral by taking y minus mu as 1 term and mu as other, we get,
mu into lambda by phi into integral from minus infinity to infinity dy by lambda square plus y minus mu the whole square plus lambda by phi into integral from minus infinity to infinity, y minus mu divided by lambda square plus y minus mu the whole square into dy
by taking y minus mu as z, and substituting the integral we get,
mu plus 1 by phi into integral from minus infinity to infinity, z divided by lambda square plus z square dz

Although the integral from minus infinity to infinity z divided by lambda square plus z square dz is not completely convergent, that is as limit n tends to infinity and n dash tends to infinity integral from minus n to n dash z divided by lambda square plus z square dz, does not exist. Its principle value namely, limit n tends to infinity integral from minus n to n, z divided by lambda square plus z square dz, exists and is equal to zero. Thus, in general sense the mean of Cauchy distribution does not exist.

However, if we conventionally agree to assume that the mean of Cauchy distribution exists (by taking the principal value), then it is located at x is equal to mu. Then, the probability curve is symmetrical about the point x is equal to mu. Hence, for this distribution, the mean, median and mode coincide at the point x is equal to mu.
Now, consider the variance,
Mu 2 is equal to Expectation of Y minus mu the whole square
Is equal to integral from minus infinity to infinity, y minus mu the whole square into f of y dy
Is equal to lambda by phi into y minus mu the whole square divided by lambda square plus y minus mu the whole square, dy.
Which does not exist since the integral is not convergent. Thus, in general, for Cauchy Distribution, the moment mu r (for r greater than or equal to 2) does not exist.

4. Logistic Distribution - MGF, Moments, Skewness and Kurtosis

Now, let us consider the logistic distribution.
Before we start the discussion about logistic distribution, let us prove the following two results, which are made use in defining the distribution and then in subsequent properties of the distribution.

Show that tan h x is equal to 1 minus e power minus 2 into x whole power minus 1
To prove this result consider,
Tan h x is equal to sine h x divided by cos h x
Is equal to 1 minus e power minus 2 into x whole divided by 1 plus e power minus 2 into x
Implies, 1 plus tan h x is equal to 1 plus (1 minus e power minus 2 into x whole divided by 1 plus e power minus 2 into x)
Simplifying by taking common denominator we get,
2 divided by 1 plus e power minus 2 into x
Implies, half into 1 plus tan h x is equal to 1 plus e power minus 2 into x whole power minus 1.

Consider the 2^{nd} result.
x into cosec x is equal to 1 plus x square divided by 6 plus 7 by 360 into x power 4 plus etc.
To prove this result, consider
x into cosec x is equal to x divided by sin x (since cosec x is equal to 1 by sin x).
Using the expansion of sin x we get,
x divided by x minus x cube by 3 factorial plus x power 5 by 5 factorial minus x power 7 by 7 factorial plus etc
Taking x common from the denominator and cancelling with the numerator and then simplifying, we get,
1 minus of x square divided by 6 minus x power 4 divided by 120 plus x power 6 divided by 7 factorial minus etc whole power minus 1.

Using the expansion of geometric series with infinite terms, we get,
1 plus x square divided by 6 minus x power 4 divided by 120 plus etc plus x square divided by 6 minus x power 4 divided by 120 plus etc whole square plus etc.
Is equal to 1 plus x square divided by 6 plus x power 4 into 1 by 36 minus 1 by 120 plus etc
Is equal to 1 plus x square divided by 6 plus 7 divided by 360 into x power 4 plus etc.

A continuous random variable X is said to have logistic distribution with parameters alpha and beta, if its distribution function is of the form,
FX of (x) is equal to [1 plus exponential of {minus (x minus alpha) divided by beta}] whole power minus 1, where beta is greater than zero, which is same as, half into 1 plus tan h of half into x minus alpha divided by beta, where beta is greater than zero.
The probability density function of logistic distribution with parameters alpha and beta is given by differentiating the distribution function with respect to x.

f of x is equal to d by dx of F of x

is equal to 1 by beta into 1 plus exponential of minus of x minus alpha divided by beta whole power minus 2 into exponential of minus of x minus alpha divided by beta
is equal to 1 by 4 into beta into secant h square of half into x minus alpha whole divided by beta.
The probability density function of standard logistic variate Y is equal to X minus alpha divided by beta is given by,
G Y of y is equal to f of x into modulus of dx by dy
Is equal to e power minus y into 1 plus e power minus y whole power minus 1, where y lies between minus infinity to infinity.
Is equal to 1 divided by 4 into secant h square into half into y.

The distribution function of Y is,
G Y of y is equal to 1 plus e power minus y whole power minus 1, where y lies between minus infinity and infinity.
Logistic distribution is extensively used as growth function in population, demographic studies and in time series analysis. Theoretically, Logistic distribution can be obtained as:

- Limiting distribution (as n tends to infinity) of the standardized mid-range, (or average of the smallest and the largest sample observations) in random samples of size n
- A mixture of extreme value distributions

Now, let us find the moment generating function of the distribution.
The moment generating function of standard Logistic variate Y is given by,
M Y of t is equal to expectation of e power t into Y
Is equal to integral from minus infinity to infinity, e power t into y into f of y dy
Is equal to integral from minus infinity to infinity, e power t into y into e power minus y into 1 plus e power minus y whole power minus 2 dy
Is equal to integral from minus infinity to infinity, e power t into y into e power minus y into 1 plus e power y divided by e power y whole power minus 2 dy
Is equal to integral from minus infinity to infinity, e power t into y into e power y into 1 plus e power y whole power minus 2 dy

Put z is equal to 1 plus e power y whole power minus 1
Implies, e power y is equal to 1 by z minus 1
Is equal to 1 minus z whole divided by z.
As y is equal to minus infinity, z is equal to 1 and as y is equal to infinity, z is equal to zero.
Therefore, M Y of t is equal to integral from 1 to zero, 1 minus z divided by z whole power t minus dz
Integral from 1 to zero is same as minus of integral from zero to 1.
Hence we get, integral from zero to 1 z power minus t into 1 minus z whole power t dz, which is beta integral of first kind and
Is equal to beta of 1 minus t and 1 plus t, where 1 minus t is greater than zero.
Writing beta function in terms of gamma, we get
Gamma 1 minus t into gamma 1 plus t divided by gamma 2
Which is equal to Gamma 1 minus t into gamma 1 plus t
Is equal to phi into t into cosecant phi t, where t is less than 1
Is equal to 1 plus phi square into t square divided by 6 plus 7 divided by 360 into phi power 4 into t power 4 plus etc.

Now, let us find moments of the distribution from the above moment generating function.

Mu 1 dash is equal to coefficient of t in moment generating function is equal to zero implies Mean is equal to zero
Since mu 1 dash is equal to zero, all the moments about mean and about origin are same.
Mu 2 dash is equal to mu 2 is equal to coefficient of t square divided by 2 factorial is equal to phi square divided by 3
Mu 3 dash is equal to mu 3 is equal to coefficient of t cube divided by 3 factorial is equal to zero
Mu 4 dash is equal to mu 4 is equal to coefficient of t power 4 divided by 4 factorial is equal to 7 into phi power 4 divided by 15
Therefore, coefficients of skewness and kurtosis are given by,
Beta 1 is equal to mu 3 square divided by mu 2 cube is equal to zero, Beta 2 is equal to mu 4 divided by mu 2 square is equal to 4.2
Hence, standard Logistic distribution is symmetric and has leptokurtic curve.

5. Mean and Variance

Now, consider some of the remarks.
We have g of (y) is equal to e power minus y into 1 plus e power minus y whole power minus 2
Writing e power minus y as 1 by e power y, we get,
E power minus y into 1 plus 1 by e power y whole power minus 2
By taking common denominator and simplifying, we get
E power y into 1 plus e power y whole power minus 2, which is equal to g of minus y
Implies the probability curve of Y is symmetric about the line y is equal to zero. Since probability density function g of (y) is symmetric about the origin (y is equal to zero), all odd order moments about origin is zero, that is mu 2 r plus 1 dash is equal to Expectation of Y power 2 r plus 1 is equal to zero for r is equal to zero, 1, 2, etc.

In particular, Mean mu 1 dash is equal to zero.
Therefore, mu r dash is equal to r^{th} moment about origin is equal to r^{th} moment about mean is equal to mu r
Implies, mu 2r plus 1 is equal to mu 2r plus 1 dash is equal to zero. That is all odd order moments about mean of the standard logistic distribution are zero.
In particular, mu 3 is equal to zero implies, beta 1 is equal to zero.

The mean and variance of the logistic Variable (X) with parameters alpha and beta are obtained as follows:
We know that Y is equal to (X minus alpha) divided by beta implies, X is equal to alpha plus beta into Y
Mean, Expectation of (X) is equal to Expectation of (alpha plus beta into Y) is equal to alpha plus beta into Expectation of Y is equal to alpha
Variance of (X) is equal to Variance of (alpha plus beta into Y) is equal to beta square into Variance of (Y) is equal to beta square into phi square divided by 3

We have G of y is equal to 1 plus e power minus y whole power minus 1
Is equal to 1 plus e power y divided by e power y whole power minus 1
Is equal to e power y divided by 1 plus e power y
Implies 1 minus G of y is equal to 1 minus e power y divided by 1 plus e power y is equal to 1 divided by 1 plus e power y
Therefore, G of y into 1 minus G of y is equal to e power y divided by 1 plus e power y into 1 divided by 1 plus e power y is equal to e power y divided by 1 plus e power y whole square is equal to g of y
Also G of y divided by 1 minus G of y is equal to e power y
Implies, y is equal to log G of y divided by 1 minus G of y

Now, let us write the expression for mean deviation for the standard logistic distribution.
M D is equal to 2 into 1 minus half plus 1 by 3 minus 1 by 4 plus etc
Is equal to 2 into summation over i is equal to 1 to infinity, minus 1 power i minus 1 divided by i is equal to 2 into log 2 to the base e.

Here’s a summary of our learning in this session, where we understood:

- The Standard Cauchy Distribution
- The Cauchy Distribution with two parameters – median, mode and moments
- The Standard Logistic Distribution – mgf, moments, skewness and kurtosis
- The Logistic Distribution with 2 parameters – mean and variance