Frequently Asked Questions

1. Define standard Cauchy distribution.

Answer:

A random variable X is said to have a standard Cauchy distribution, if its pdf is given by,

$$f(x) = \frac{1}{\pi(1+x^2)}, -\infty < x < \infty$$

2. Define Cauchy distribution.

Answer:

Cauchy distribution with parameters λ and μ has the pdf,

$f(y) = \frac{\lambda}{\pi\left[\lambda^2 + (y-\mu)^2\right]}, -\infty < y < \infty; \lambda > 0$ and we write X~C(λ,μ)

3. Obtain median of Cauchy distribution.

Answer:

Let Y~C(λ,μ) and If M is the median of the distribution then

$$\int_{-\infty}^{M} f(y)dy = \frac{1}{2} = \int_{M}^{\infty} f(y)dy$$

We can solve any of the integral. For simplicity let us consider the 2nd integral,

$$\int_{M}^{\infty} f(y)dy = 1/2$$

The range of the above integral can be split into two parts, namely, from M to μ and μ to ∞.

Hence we get,

$$\int_{M}^{\mu} f(y)dy + \int_{\mu}^{\infty} f(y)dy = \frac{1}{2} \text{ ------(1)}$$

Now consider the 2nd integral from 1.

$$i.e., \int_{\mu}^{\infty} f(y)dy = \int_{\mu}^{\infty} \frac{\lambda}{\pi\left[\lambda^2 + (y-\mu)^2\right]} dy$$

Substitute, x=(y-μ)/λ. As y=∞, x= ∞ and as y=μ, x=0. Also dx=λdy

$$\therefore \int_{\mu}^{\infty} f(y)dy = \frac{1}{\pi}\int_{0}^{\infty} \frac{1}{1+x^2} dx = \frac{1}{2}$$

Substituting in 1, we get,

$$\int_{M}^{\mu} f(y)dy + \frac{1}{2} = \frac{1}{2} \Rightarrow \int_{M}^{\mu} f(y)dy = 0$$

⇒M=μ.

4. Derive mode of the Cauchy distribution.

Answer:

Mode is the value for which f(y) is maximum.

We can find the maximum by differentiating the probability density function with respect of y, then equating it zero and solving form y. Then obtain the 2nd derivative which should be negative.

If Y~C(λ,μ), then $f(y) = \dfrac{\lambda}{\pi\left[\lambda^2 + (y-\mu)^2\right]}, -\infty < y < \infty; \lambda > 0$

To simplify the differentiating, let us work with logarithm.
By taking log on both the sides, we get,
Log f(y) = log λ-logπ-log [λ^2-(y-μ)^2]
On differentiating we get,
f'(y)/f(y)=2(y-μ)/ [λ^2-(y-μ)^2]
f'(y)=0 ⇒y= μ
If we find 2nd derivative, at y= μ, it is negative.
Hence y= μ is the mode of the distribution.

5. Find mean of the Cauchy distribution.

Answer:

If Y~C(λ,μ) then mean of the distribution is given by,

$$E(Y) = \int_{-\infty}^{\infty} y f(y) dy = \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{y}{\lambda^2 + (y-\mu)^2} dy$$
$$= \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{(y-\mu)+\mu}{\lambda^2 + (y-\mu)^2} dy = \mu\frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{dy}{\lambda^2 + (y-\mu)^2} + \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{(y-\mu)}{\lambda^2 + (y-\mu)^2} dy$$
$$= \mu + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{z}{\lambda^2 + z^2} dz$$

Although the integral $\int_{-\infty}^{\infty} \frac{z}{\lambda^2 + z^2} dz$ is not completely convergent, i.e. $\lim_{\substack{n\to\infty \\ n'\to\infty}} \left(\int_{-n}^{n'} \frac{z}{\lambda^2 + z^2} dz \right)$, does not exist, its principle value viz., $\lim_{n\to\infty} \left(\int_{-n}^{n} \frac{z}{\lambda^2 + z^2} dz \right)$, exists and is equal to zero. Thus in general sense the mean of Cauchy distribution does not exist.

But, if we conventionally agree to assume that the mean of Cauchy distribution exists (by taking the principal value), then it is located at x=μ. Also obviously, the probability curve is symmetrical about the point x=μ. Hence for this distribution, the mean, median and mode coincide at the point x=μ.

6. Obtain Variance of the Cauchy distribution.

Answer:

Now, consider the variance,

$\mu_2 = E(Y-\mu)^2 = \int_{-\infty}^{\infty} (y-\mu)^2 f(y) dy = \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{(y-\mu)^2}{\lambda^2 + (y-\mu)^2} dy$, which does not exist since the integral is not convergent. Thus, Variance of the distribution does not exist.

7. Show that tanh x = $(1-e^{-2x})^{-1}$.

Answer:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{1-e^{-2x}}{1+e^{-2x}}$$
$$\Rightarrow 1 + \tanh(x) = 1 + \frac{1-e^{-2x}}{1+e^{-2x}} = \frac{2}{1+e^{-2x}} \Rightarrow \frac{1}{2}[1+\tanh(x)] = (1+e^{-2x})^{-1}$$

8. Find Cumulant generating function.

Answer:

We know that Cumulant generating function is given by
$K_X(t)=\log(M_X(t))=\log[1-(t/\theta)]^{-1}$
$=-1\log[1-(t/\theta)]$

$$=-1\left[t/\theta-\frac{(t/\theta)^2}{2}-\frac{(t/\theta)^3}{3}-\frac{(t/\theta)^4}{4}-\dots\right]=t/\theta+\frac{(t/\theta)^2}{2}+\frac{(t/\theta)^3}{3}+\frac{(t/\theta)^4}{4}+\dots$$

9. Show that x cosec x = 1 + $x^2/6$ + $(7/360)x^4$+….

Answer:

x cosec x = x/(sin x)

$$=\frac{x}{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots}=\left[1-\left(\frac{x^2}{6}-\frac{x^4}{120}+\frac{x^6}{7!}-\dots\right)\right]^{-1}$$

Using the expansion of geometric series with infinite terms, we get,

$$=1+\left(\frac{x^2}{6}-\frac{x^4}{120}+\dots\right)+\left(\frac{x^2}{6}-\frac{x^4}{120}+\dots\right)^2+\dots=1+\frac{x^2}{6}+x^4\left(\frac{1}{36}-\frac{1}{120}\right)+\dots=1+\frac{x^2}{6}+\frac{7}{360}x^4+\dots$$

10. Define Logistic distribution.

Answer:

A continuous random variable X is said to have logistic distribution with parameters α and β, if its distribution function is of the form $F_X(x)=[1+\exp\{-(x-\alpha)/\beta\}]^{-1}$, β>0

$$=\frac{1}{2}\left[1+\tanh\left\{\frac{1}{2}\frac{(x-\alpha)}{\beta}\right\}\right];\beta>0$$

11. Obtain the pdf of logistic distribution.

Answer:

The pdf of logistic distribution with parameters α and β is given by differentiating the distribution function with respect to x.

$$f(x)=\frac{d}{dx}[F(x)]$$

$$=\frac{1}{\beta}[1+\exp(-(x-\alpha)/\beta]^{-2}[\exp(-(x-\alpha)/\beta]=\frac{1}{4\beta}\operatorname{sec}h^2\left\{\frac{1}{2}(x-\alpha)/\beta\right\}$$

12. How to obtain logistic distribution theoretically?

Answer:

Theoretically, Logistic distribution can be obtained as

- Limiting distribution (as n→∞) of the standardized mid-range, (average) of the smallest and the largest sample observations), in random samples of size n.
- A mixture of extreme value distributions.

13. Obtain mgf of standard logistic variate.

Answer:

The mgf of standard Logistic variate Y is given by,

$$M_Y(t)=E(e^{tY})=\int_{-\infty}^{\infty}e^{ty}f(y)dy$$

$$=\int_{-\infty}^{\infty}e^{ty}e^{-y}(1+e^{-y})^{-2}dy=\int_{-\infty}^{\infty}e^{ty}e^{-y}\left(\frac{1+e^y}{e^y}\right)^{-2}dy=\int_{-\infty}^{\infty}e^{ty}e^{y}(1+e^{y})^{-2}dy$$

Put $z=(1+e^y)^{-1} \Rightarrow e^y=1/z - 1 = (1-z)/z$.

As $y=-\infty$, $z=1$ and as $y=\infty$, $z=0$

$$M_Y(t) = \int_1^0 \left(\frac{1-z}{z}\right)^t (-dz)$$

$$= \int_0^1 z^{-t}(1-z)^t (dz)$$

$$= B(1-t, 1+t) = \frac{\Gamma(1-t)\Gamma(1+t)}{\Gamma 2}$$

$$= \Gamma(1-t)\Gamma(1+t) = \pi t.\cos ec(\pi t); t < 1$$

$$= 1 + \frac{\pi^2 t^2}{6} + \frac{7}{360}\pi^4 t^4 + \ldots$$

14. Obtain first 4 moments of standard logistic distribution and hence find coefficient of skewness and kurtosis.

Answer:

μ_1'=coefficient of t in mgf = 0 $\Rightarrow$ Mean = 0

Since $\mu_1'=0$, all the moments about mean and about origin are same.

$\mu_2' = \mu_2$=coefficient of $t^2/2! = \pi^2/3$

$\mu_3' = \mu_3$=coefficient of $t^3/3! = 0$

$\mu_4' = \mu_4$=coefficient of $t^4/4! = 7\pi^4/15$

Therefore coefficients of skewness and kurtosis are given by,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0, \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{7 \times 9}{15} = 4.2$$

Hence standard Logistic distribution is symmetric and has leptokurtic curve.

15. Obtain mean and variance of the logistic Variable (X) with parameters α and β.

Answer:

The mean and variance of the logistic Variable (X) with parameters α and β are obtained as follows:

We know that $Y=(X-\alpha)/\beta \Rightarrow X= \alpha+\beta Y$

$E(X)=E(\alpha+\beta Y)= \alpha+\beta E(Y)= \alpha$

$Var(X)=Var(\alpha+\beta Y)= \beta^2 Var(Y)= \beta^2 \pi^2/3$