1. Gamma Distribution

Welcome to the series of E-learning modules on Gamma and Beta distributions.
Here, we will discuss the various measures of gamma distribution, namely -mean, variance, mgf and additive property. Beta distribution of 1st and 2nd kind is also discussed with their various measures. Relationship between different distributions is also given.

By the end of this session, you will be able to:

- Explain gamma distribution – mean, variance, mgf and additive property
- Explain beta distribution of first kind – mean, variance, coefficient of skewness and kurtosis
- Explain beta distribution of second kind – mean and variance
- Explain the relation between beta distribution and other distributions

First, let us discuss about gamma distribution.
A random variable X is said to follow gamma distribution with parameters alpha and beta, if its probability density function is given by,
f of x is equal to beta power alpha divided by gamma alpha into e power minus beta into x into x power alpha minus 1, where x is greater than zero.

Since above is a probability density function, integral over x, f of x d x is equal to 1.
Implies, integral from zero to infinity, f of x dx is equal to 1
Implies, integral from zero to infinity e power minus beta into x into x power alpha minus 1 dx
Is equal to gamma alpha divided by beta power alpha

Let us find mean of the distribution.
If X follows gamma distribution with parameters, alpha and beta then, mean mu 1 dash is equal to expectation of X is equal to integral from zero to infinity, x into f of x dx
Is equal to beta power alpha divided by gamma alpha into integral from zero to infinity, x into e power minus beta into x into x power alpha minus 1 dx
Is equal to beta power alpha divided by gamma alpha into integral e power minus beta into x into x power alpha plus 1 minus 1 dx
Is equal to beta power alpha divided by gamma alpha into gamma alpha plus 1 divided by beta power alpha plus 1
Is equal to alpha divided by beta

Variance of the distribution is given by, Variance of X is equal to mu two is equal to mu 2 dash minus mu 1 dash the whole square
First, let us find mu 2 dash is equal to expectation of X square
Is equal to integral from zero to infinity, x square into f of x dx
Is equal to beta power alpha divided by gamma alpha into integral from zero to infinity, x square into e power minus beta into x into x power alpha minus 1 dx
Is equal to beta power alpha divided by gamma alpha into integral from zero to infinity, e power minus beta into x into x power alpha plus 2 minus 1 dx
Is equal to beta power alpha divided by gamma alpha into gamma alpha plus 2 divided by beta power alpha plus 2

Is equal to alpha into alpha plus 1 divided by beta square.
Hence
Mu 2 is equal to variance of X is equal to alpha into alpha plus 1 divided by beta square minus alpha by beta the whole square
Is equal to alpha divided by beta square

2. Moment Generating Function of Gamma Distribution

Let us obtain moment generating function.
If X follows Gamma distribution with parameters (alpha and beta) then moment generating function is given by,
M X of t is equal to Expectation of e power t into X
Is equal to integral from zero to infinity e power t into x into f of x dx
Is equal to beta power alpha by gamma alpha into integral from zero to infinity, e power t into x into e power minus beta into x into x power alpha minus 1 dx
Is equal to beta power alpha divided by gamma alpha into integral from zero to infinity e power minus x into beta minus t into x power alpha minus 1 dx
Is equal to beta power alpha divided by gamma alpha into gamma alpha divided by beta minus t whole power alpha, which is equal to 1 divided by 1 minus t by beta whole power alpha

The sum of independent gamma variates is also a gamma variate. More precisely, if X1, X2, … Xn are independent variables, such that Xi follows Gamma distribution with parameters (alpha I and beta) then,
X1 plus X2 plus up to plus X n is also gamma variate with parameter alpha 1 plus alpha 2 plus up to alpha n
That is, summation over i is equal to1 to n, Xi follows gamma distribution with parameter summation over i is equal to 1 to n alpha I and beta
Let us prove this result using moment generating function
Since Xi follows Gamma distribution with parameters (alpha i and beta), the moment generating function MXi of t is equal to 1 minus t by beta whole power minus alpha i

The moment generating function of the sum X1 plus X2 plus up to plus X n is given by,
M sum X1 plus X2 plus up to plus X n is equal to M X1 of t into M X2 of t up to M Xn of t (since X1, X2 up to Xn are independent).
Is equal to 1 minus t by beta power minus alpha 1 into 1 minus t by beta power minus alpha 2 up to1 minus t by beta power minus alpha n
Is equal to 1 minus t by beta power minus alpha 1 plus alpha 2 plus up to alpha n
Is equal to 1 minus t by beta power minus summation over i is equal to 1 to n alpha i, which is the moment generating function of gamma variate with parameters summation alpha I and beta. Hence by uniqueness theorem of moment generating function, summation over i is equal to1 to n Xi follows gamma distribution with parameter summation over i is equal to 1 to n alpha i and beta

Show that if Xi where i is equal to 1, 2, up to n are independently identically distributed exponential random variables with parameter beta, then summation Xi follows gamma distribution with parameters n and beta.
We show this result by making use of moment generating function of gamma distribution.

If Xi follows gamma distribution with parameter beta, then the moment generating function is

given by,
MXi of t is equal to 1 minus t divided by beta whole power minus 1
Now, consider the moment generating function of the sum X1 plus X2 plus up to plus Xn
That is, M X1 plus X2 plus up to plus Xn of t is equal to M sum X1 plus X2 plus up to plus Xn is equal to M X1 of t into M X2 of t up to MXn of t
(since X1, X2, up to Xn are independent)

Is equal to 1 minus t divided by beta power minus 1 into 1 minus t divided by beta power minus 1 up to 1 minus t divided by beta power minus 1 (n times)
Is equal to 1 minus t divided by beta whole power minus n, which is the moment generating function of gamma distribution with parameters n and beta. Hence, by uniqueness theorem of moment generating function, summation Xi follows gamma distribution with parameters n and beta.

3. Beta Distribution of First Kind

Now, let us discuss beta distribution of First kind.
A random variable X is said to follow beta distribution of first kind with parameters m and n if its probability density function is given by,
f of x is equal to 1 divided by beta of m n into x power m minus 1 into 1 minus x power n minus 1, where zero less than x less than 1
Where, Beta of (m, n) is known as beta function and is given by,
Beta of m, n is equal to gamma m into gamma n divided by gamma m plus n

Let us consider some remarks.
As f of (x) is a probability density function, we can write
Integral from zero to 1 f of x dx is equal to 1
Implies, integral from zero to 1 x power m minus 1 into 1 minus x power n minus 1 dx is equal to Beta of m, n
We can write this distribution as,
X follows beta 1 of m, n, and read as X follows beta distribution of first kind with parameters m and n.

2. In particular, if we take m is equal to 1 and n is equal to 1 in the probability density function of beta distribution of first kind, we get, f of x is equal to 1 divide by beta of 1, 1 into x power 1 minus 1 into 1 minus x power 1 minus 1, which is equal to 1, where x lies between zero and 1, which is the probability density function of uniform distribution on (0,1).
3. If X follows beta distribution of first kind with parameters m and n then, 1 minus X follows beta distribution of first kind waith parameters n and m.

If X follows beta distribution of first kind with parameters m and n, then the mean is given by,
Mu 1 dash is equal to expectation of X is equal to integral from zero to 1 x into f of x d x
Is equal to 1 divided by beta of m n into integral from zero to 1, x into x power m minus 1 into 1 minus x power n minus 1 dx
Is equal to 1 divided by beta of m n into integral from zero to 1, x power m plus 1 minus 1 into 1 minus x power n minus 1 dx
Is equal to 1 divided by beta of m n into beta of m plus 1, n
Writing beta functions in terms of gamma and simplifying we get, mu 1 dash is equal to m divided by m plus n

If X follows beta distribution of first kind with parameters m and n, the mean is given by,
Variance of X is equal to mu to is equal to mu 2 dash minus mu 1 dash the whole square
First, let us find mu 2 dash is equal to expectation of X square
Is equal to integral from zero to 1 x square into f of x dx
Is equal to 1 divided by beta of m n into integral from zero to 1, x square into x power m minus 1 into 1 minus x power n minus 1 dx
Is equal to 1 divided by beta of m n into integral from zero to 1, x power m plus 2 minus 1 into 1 minus x power n minus 1 dx
Is equal to 1 divided by beta of m n into beta of m plus 2, n

4. Simplification of Beta Functions in Terms of Gamma

Writing beta functions in terms of gamma and then simplifying, we get,
Mu 2 dash is equal to m into m plus 1 divided by m plus n plus 1 into m plus n
Therefore, the variance mu 2 is equal to m into m plus 1 divided by m plus n plus 1 into m plus n minus m by m plus n whole square
On simplifying we get,
m into n divided by m plus n plus 1 into m plus n the whole square

In general, rth raw moment is given by, mu r dash is equal to Expectation of X power r is equal to integral from zero to 1, x power r into f of x dx
Is equal 1 divided by beta of m, n into integral from zero to 1 x power r into x power minus into 1 minus x power n minus 1 dx
Is equal to 1 divided by beta of m, n into integral from zero to 1, x power m plus r minus 1 into 1 minus x power n minus 1 dx
Is equal to 1 divided by beta of m, n into beta of m plus r, n
Is equal to gamma (m plus n) divided by gamma m gamma n into gamma (m plus r) into gamma n divided by gamma (m+r+n)
Substituting r is equal to 1 we get, mean and r is equal to 2, 2nd raw moment and in turn variance of the distribution .

Similarly, we have mu 3 is equal to mu 3 dash minus 3 into mu 2 dash into mu 1 dash plus 2 mu 1 dash cube
Is equal to 2 into m into n into n minus m whole divided by, m plus n cube into m plus n plus 1 into m plus n plus 2
And mu 4 is equal to mu 4 dash minus 4 into mu 3 dash into mu 1 dash plus 6 into mu 2 dash into mu 1 dash square minus 3 into mu 1 dash power 4
Is equal to 3 into m into n into (m n into m plus n minus 6 plus 2 into m plus n square divided by m plus n power 4 into m plus n plus 1 into m plus n plus 2 into m plus n plus 3

Hence, coefficient of skewness is given by,
Beta 1 is equal to mu 3 square divided by mu 2 cube
Is equal to 4 into n minus m the whole square into m plus n plus 1 divided by m into n into m plus n plus 2 the whole square
Coefficient of kurtosis is given by,
Beta 2 is equal to mu 4 divided by mu 2 square
Is equal to 3 into m plus n plus 1 into m into n into m plus n minus 6 plus 2 into m plus n the whole square whole divided by m into n into m plus n plus 2 into m plus n plus 3

5. Beta Distribution of Second Kind

Now, let us consider the beta distribution of 2nd kind.
A random variable X is said to follow beta distribution of 2nd kind with parameters m and n if its probability density function is given by,
f of x is equal to 1 divided by beta of m, n into x power m plus 1 divided by 1 plus x power m plus n, where x is greater than zero
and we write X follows Beta 2 of m, n, read as X follows beta distribution of 2nd kind with parameters m and n
Since f of x is a probability density function, we can write
Integral from zero to infinity, f of x dx is equal to 1
Implies, integral from zero to infinity, x power m plus 1 divided by 1 plus x power m plus n dx is equal to beta of m, n

Now, let us find rth raw moment
In general, rth raw moment is given by,
Mu r dash is equal to expectation of X power r
Is equal to integral from zero to infinity x power r into f of x dx
Is equal to 1 divided by beta of m, n, into integral over zero to infinity x power r into x power m minus 1 divided by 1 plus x power m plus n dx
Is equal to 1 divided by beta of m, n integral over zero to infinity, x power m plus r minus 1 divided by 1 plus x power m plus r plus n minus r, dx
Is equal to beta of m plus r, n minus r divided by beta of m, n
Is equal to gamma (m plus r) into gamma (n minus r) divided by gamma m gamma n
With this general expression let us find mean and variance

In the above general expression, if we put r is equal to 1 we get mean.
That is, mu 1 dash is equal to gamma m plus 1 into gamma n minus 1 divided by gamma m into gamma n
On expanding gamma terms and simplifying, we get, m divided by n minus 1

If we put r is equal to 2, we get,
Mu 2 dash is equal to gamma m plus 2 into gamma n minus 2 divided by gamma m into gamma n
On expanding gamma terms and simplifying, we get, m into m plus 1 divided by n minus 1 into n minus 2
Hence, variance is given by,
Mu 2 is equal to m into m plus 1 divided by n minus 1 into n minus 2 minus m divided by n minus 1 the whole square
On simplifying, we get,
m into m plus n minus 1 divided by n minus 1 the square into n minus 2

Now, let us discuss the relation between two kinds of beta distribution. Here, one kind can be transformed into other by taking some transformation.
Beta distribution of second kind is transformed to beta distribution of first kind by the transformation 1 plus x is equal to 1 divided by y or y is equal to 1 divided by (1 plus x)
Thus, if X follows beta distribution of 2nd kind with parameters (m, and n) then, Y is a beta

distribution of 1st kind variable with parameters (m and n)

If X and Y are two independent gamma variates, then
Ratio of two variables, X divided by Y is a beta variate of 2nd kind
The ratio X divided by (X plus Y) follows beta distribution of 1st kind.
(Proof of these results will be discussed in transformation of random variables)

Here's a summary of our learning in this session, where we understood:

- The Gamma distribution – mean, variance, mgf and additive property
- The Beta distribution of first kind – mean, variance, coefficient of skewness and kurtosis
- The Beta distribution of 2nd kind – mean and variance
- The relation between beta distribution and other distributions