Frequently Asked Questions

1. Define gamma distribution.

Answer:

A random variable X is said to follow gamma distribution with parameters α, β if its pdf is given by, $f(x) = \frac{\beta^{\alpha}}{\Gamma\alpha} e^{-\beta x} x^{\alpha-1}, x > 0$

2. Define beta distribution of 1st kind.

Answer:

A random variable X is said to follow beta distribution of first kind with parameters m and n if its pdf is given by, $f(x) = \frac{1}{B(m,n)} x^{m-1}(1-x)^{n-1}, 0 < x < 1$

3. Define beta distribution of 2nd kind.

Answer:

A random variable X is said to follow beta distribution of 2^{nd} kind with parameters m and n if its pdf is given by $f(x) = \frac{1}{B(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}}, x > 0$

4. In beta distribution of first kind, if parameters m and n are equal to 1, then identify the resulting distribution.

Answer:

If we take m=1 and n=1 in the pdf of beta distribution of first kind, we get, $f(x) = 1, 0 < x < 1$, which is the pdf of uniform distribution on (0,1).

5. How to obtain beta distribution of first kind from beta distribution of first kind?

Answer:

Beta distribution of second kind is transformed to beta distribution of first kind by the transformation, 1+x=1/y

Or y = 1/(1+x)

Thus, if X~ β_2(m, n) the Y is a β_1(m, n)

6. Find mean of gamma distribution.

Answer

If X ~ Gamma (α, β) then mean μ_1'

$$E(X) = \int_0^{\infty} x f(x) dx = \frac{\beta^{\alpha}}{\Gamma\alpha} \int_0^{\infty} x e^{-\beta x} x^{\alpha-1} dx = \frac{\beta^{\alpha}}{\Gamma\alpha} \int_0^{\infty} e^{-\beta x} x^{(\alpha+1)-1} dx = \frac{\beta^{\alpha}}{\Gamma\alpha} \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} = \frac{\alpha}{\beta}$$

7. Obtain variance of gamma distribution.

Answer:

Variance of the distribution is given by, V(X)= $\mu_2 = \mu_2' - [\mu_1']^2$

First let us find μ_2'

$$\mu_2' = E(X^2) = \int_0^{\infty} x^2 f(x) dx$$

$$= \frac{\beta^{\alpha}}{\Gamma\alpha} \int_0^{\infty} x^2 e^{-\beta x} x^{\alpha-1} dx = \frac{\beta^{\alpha}}{\Gamma\alpha} \int_0^{\infty} e^{-\beta x} x^{(\alpha+2)-1} dx = \frac{\beta^{\alpha}}{\Gamma\alpha} \frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}} = \frac{\alpha(\alpha+1)}{\beta^2}$$

$$E(X)=\int_0^\infty xf(x)dx\ =\frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty xe^{-\beta x}x^{\alpha-1}dx=\frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty e^{-\beta x}x^{(\alpha+1)-1}dx=\frac{\beta^\alpha}{\Gamma\alpha}\frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}}=\frac{\alpha}{\beta}$$

Hence $\mu_2=V(X)=\dfrac{\alpha(\alpha+1)}{\beta^2}-\left(\dfrac{\alpha}{\beta}\right)^2=\dfrac{\alpha}{\beta^2}$

8. Obtain mgf of gamma distribution.

Answer:

If X~ Gamma(α, β) then mgf of the distribution is given by

$$M_X(t)=E(e^{tX})=\int_0^\infty e^{tx}f(x)dx=\frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty e^{tx}e^{-\beta x}x^{\alpha-1}dx$$

$$=\frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty e^{-x(\beta-t)}x^{\alpha-1}dx=\frac{\beta^\alpha}{\Gamma\alpha}\frac{\Gamma\alpha}{(\beta-t)^\alpha}=\frac{1}{(1-t/\beta)^\alpha}$$

9. State and prove additive property of gamma distribution.

Answer:

The sum of independent gamma variates is also a gamma variate. More precisely, if X_1, X_2,… X_n are independent variables, such that X_i ~ Gamma(α_i, β) then $X_1+X_2+\ldots+X_n$ is also gamma variate with parameter $\alpha_1+\alpha_2+\ldots+\alpha_n$.

i.e., $\sum_{i=1}^{n} X_i \sim Gamma(\sum_{i=1}^{n}\alpha_i,\beta)$

Proof

Since X_i ~ Gamma(α_i, β), $M_{X_i}(t)=(1-t/\beta)^{-\alpha_i}$

The mgf of the sum $X_1+X_2+\ldots+X_n$ is given by,

$$M_{X_1+X_2+\ldots+X_n}(t)=M_{X_1}(t)M_{X_2}(t)\ldots M_{X_n}(t)$$

(since X_1, X_2, … X_n are independent)

$$=(1-t/\beta)^{-\alpha_1}(1-t/\beta)^{-\alpha_2}\ldots(1-t/\beta)^{-\alpha_n}$$

$$=(1-t/\beta)^{-(\alpha_1+\alpha_2+\ldots\alpha_n)}=(1-t/\beta)^{-\sum_{i=1}^{n}\alpha_i}$$

Which is the mgf of gamma variate with parameter $\Sigma\alpha_i$, and β. Hence $\sum_{i=1}^{n} X_i \sim Gamma(\sum_{i=1}^{n}\alpha_i,\beta)$

10. Obtain an expression for r^{th} raw moment of beta distribution of first kind.

Answer:

In general, r^{th} raw moment is given by,

$$\mu_r^{'}=E(X^r)=\int_0^1 x^r.f(x)dx=\frac{1}{B(m,n)}\int_0^1 x^r.x^{m-1}(1-x)^{n-1}dx$$

$$=\frac{1}{B(m,n)}\int_0^1 x^{(m+r)-1}(1-x)^{n-1}dx=\frac{1}{B(m,n)}B(m+r,n)=\frac{\Gamma(m+n)}{\Gamma m\Gamma n}\frac{\Gamma(m+r)\Gamma n}{\Gamma(m+r+n)}$$

11. Obtain the expression for r^{th} raw moment of beta distribution of 2nd kind and hence find mean and variance.

Answer:

In general r^{th} raw moment is given by,

$$\mu_r' = E(X^r) = \int_0^\infty x^r.f(x)dx = \frac{1}{B(m,n)}\int_0^\infty \frac{x^r.x^{(m)-1}}{(1+x)^{m+n}}dx$$

$$= \frac{1}{B(m,n)}\int_0^\infty \frac{x^{(m+r)-1}}{(1+x)^{(m+r)+(n-r)}}dx = \frac{B(m+r,n-r)}{B(m,n)} = \frac{\Gamma(m+r)\Gamma(n-r)}{\Gamma m\Gamma n}$$

In the above general expression, if we put r=1 we get mean.

$$\mu_1' = \frac{\Gamma(m+1)\Gamma(n-1)}{\Gamma m\Gamma n} = \frac{m}{n-1}$$

If we put, r=2, we get, $\mu_2' = \frac{\Gamma(m+2)\Gamma(n-2)}{\Gamma m\Gamma n} = \frac{m(m+1)}{(n-1)(n-2)}$

Hence variance is given by,

$$\mu_2 = \frac{m(m+1)}{(n-1)(n-2)} - \left(\frac{m}{(n-1)}\right)^2 = \frac{m(m+n-1)}{(n-1)^2(n-2)}$$

12. Write the relation between gamma and beta distributions?

Answer:

If X and Y are two independent gamma variates, then

- Ratio of two variables, X/Y is a beta variate of II kind.
- The ratio, X/(X+Y) follows beta distribution of I kind.

13. Write the expressions for coefficient of skewness and kurtosis for beta distribution of first kind.

Answer:

Coefficient of skewness is given by, $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{4(n-m)^2(m+n+1)}{mn(m+n+2)^2}$

Coefficient of kurtosis is given by, $\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3(m+n+1)mn(m+n-6)+2(m+n)^2}{mn(m+n+2)(m+n+3)}$

14. Define gamma function.

Answer:

If X has gamma distribution with parameters α, β then $f(x) = \frac{\beta^\alpha}{\Gamma\alpha}e^{-\beta x}x^{\alpha-1}, x > 0$

Since above is a pdf, $\int_x f(x)dx = 1 \Rightarrow \int_0^\infty f(x) = 1 \Rightarrow \int_0^\infty e^{-\beta x}x^{\alpha-1}dx = \frac{\Gamma\alpha}{\beta^\alpha}$

The integral function is known as gamma function.

15. Show that if Xi, i=1, 2, … n are iid exponential random variables with parameter β, then ΣXi~Gamma(n, β)

Answer:

X_i~exp(β), then mgf is given by, $M_{X_i}(t) = (1 - t/\beta)^{-1}$

Now consider mgf of the sum $X_1+X_2+ \dots +X_n$

$$M_{X_1+X_2+\ldots+X_n}(t) = M_{X_1}(t)M_{X_2}(t)\ldots M_{X_n}(t)$$

$$\text{s} \;= (1-t/\beta)^{-1}(1-t/\beta)^{-1}\ldots(1-t/\beta)^{-1} = (1-t/\beta)^{-n}$$

Which is mgf of gamma distribution with parameters n and β. Hence, by uniqueness theorem of mgf, ΣX_i~Gamma(n, β).