1. Introduction

Welcome to the series of E-learning module on Uniform distribution. In this module we will cover the definition of uniform distribution through probability density function, understand the distribution function, moments, moment generating functions, median and mean deviation.

By the end of this session, you will be able to:

- Describe the definition of uniform distribution through probability density function (pdf)
- Understand distribution function
- Explain Moments that is, mean and variance
- Understand Moment generating function
- Describe Median and
- Understand Mean deviation

A continuous random variable X is said to follow uniform distribution in the interval (a, b) if its probability density function is given by,
f of (x) is equal to one divided by (b minus a), zero less than x less than one, and we write X follows Uniform (a, b)
Here a and b are the parameters of the distribution.

In particular if a is equal to zero and b is equal to theta then we can write, X follows uniform(zero, theta)
If a is equal to minus theta and b is equal to theta, then X follows uniform (minus theta, theta). Hence we can define uniform distribution for any two constants a and b.

2. Remarks, Mean and Variance

Consider some remarks

- a and b, (a less than b) are the two parameters of the distribution. The distribution is called uniform distribution on (a, b) since it assumes a constant or (uniform) value over all x of (a, b)
- The distribution is also known as rectangular distribution, since the curve y is equal to f of (x) describes a rectangle over the x axis and between the ordinates x is equal to a and x is equal to b

Now let us find cumulative distribution function or distribution function.
The cumulative distribution function of the uniform distribution on (a, b) is given by,
F of x is equal to integral from a to x f of x d x
Is equal to one divided by b minus ‘a’, into integral from a to x d x
Is equal to one divided by b minus ‘a’, into x, ranges from ‘a’ to x
Is equal to x minus a divided by b minus ‘a’
Hence we can write,
F of x is equal to zero for x is less than or equal to ‘a’,
X minus ‘a’ divided by b minus ‘a’ for a less than x less than b
And one if x is greater than or equal to b

Figure ‘a’ shows the graph of pdf f(x) and figure ‘b’ shows the corresponding distribution function of the same.

Figure

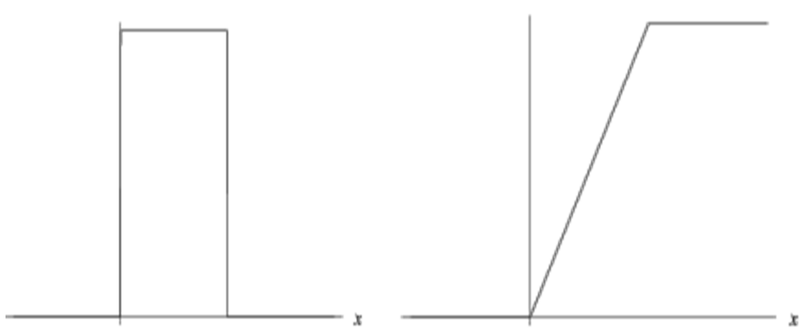

Fig a Fig b

Now let us find mean of the distribution
If X follows uniform a, b then the mean of the distribution is given by,
Expectation of X is equal to integral over x, x into f of x d x
Is equal to one divided by b minus ‘a’ into integral from ‘a’ to b, x into d x
By integrating we get, one divided by b minus ‘a’ into x square divided by two, ranges from ‘a’ to b.
Is equal to one divided by two into b minus ‘a’ into b square minus ‘a’ square.
On simplification we get b plus ‘a’ divided by two.

Now let us find variance of the distribution.
If X follows uniform a, b then the variance of the distribution is given by,
V of X is equal to Expectation of X square minus Expectation of X whole square

First let us find Expectation of X square
Expectation of X square is equal to
Integration over x, x square into f of x d x.
Is equal to one divided by b minus ‘a’ into integral from a to b x square into d x
By integrating x square, we get,
one divided by b minus ‘a’ into x cube divided by three, ranges from ‘a’ to b
Is equal to one divided by three into b minus ‘a’ into b cube minus ‘a’ cube.
Using formula for b cube minus ‘a’ cube, we get, b square plus ‘a’ into b plus b square into b minus ‘a’ divided by three into b minus ‘a’
Is equal to b square plus ‘a’ into b plus b square divided by three.

Also we have found expectation of X is equal to b plus ‘a’ divided by two
Hence, variance is given by,
Variance of X is equal to expectation of X square minus Expectation of X the whole square
By substituting we get,
b square plus ‘a’ into b plus b square divided by three minus b plus ‘a’ divided by two the whole square.
Taking common denominator as twelve, we get,
Four into b square plus ‘a’ into b plus b square minus three into b plus ‘a’ the whole square divided by twelve.
On simplification we get,
b minus ‘a’ the whole square divided by twelve.

3. Moments, Moment Generating Functions, Median and Mean Deviation

In general, moments of the uniform distribution with parameters ‘a’ and b can be obtained as follows.
Mu r dash is equal to expectation of X to the power r
Is equal to integral over x, x power r into f of x d x
Is equal to 0ne divided by b minus ‘a’ into integral from a to b x power r d x
Is equal to 0ne divided by b minus ‘a’ into x power to the ‘r’ divided by r plus one, ranges from ‘a’ to b
Is equal to 0ne divided by b minus ‘a’ into b power ‘r’ minus ‘a’ to the power ‘r’ divided by r plus 0ne.
By substituting ‘r’ is equal to 0ne and two, we can obtain expectation of X and expectation of X square and hence mean and variance.

mgf
Let us obtain the moment generating function of the distribution.
If X follows uniform a, b then Moment generating function of uniform distribution can be obtained as follows

Expectation of e to the power t into X is equal to integral over x, e to the power ‘t’ into x f of x d x.
Is equal to 0ne divided by b minus ‘a’ into integral from ‘a’ to b e to the power t into x d x
By integrating e to the power t into x, we get
0ne divided by b minus ‘a’ into e to the power t into X divided by t, ranges from a to b
Is equal to 0ne divided by t into b minus ‘a’ into e power t into b minus e to the power t into ‘a’, at t not equal to zero.

Median
Let us find the median of the distribution
If X follows uniform a, b and M is the median of the distribution, then,
Integral from ‘a’ to M f of x d x is equal to half is equal to integral from M to b f of x d x.
Hence we get M by solving any one of the integral. Hence consider
Integral from ‘a’ to M f of x d x is equal to half.
Implies, 0ne divided by b minus ‘a’ into integral from ‘a’ to M d x is equal to half.

Implies, 0ne divided by b minus ‘a’ into x ranges from ‘a’ to M is equal to half
Implies, 0ne divided by b minus ‘a’ into M minus ‘a’ is equal to half.
On simplification , we get,
M is equal to b plus ‘a’ divided by two.
Observe that for uniform distribution mean = median.

Since mean and median are same for uniform distribution, we can find mean deviation about

mean or median, which are also same.
If X follows uniform a, b then mean deviation about mean is given by,
Expectation of mod X minus Expectation of X is equal to expectation of mod X minus(b plus 'a') divided by two
Is equal to integral over x mod x minus b plus 'a' divided by two into f of x d x
Let us take t is equal to x minus b plus 'a' divided by two so that, when x is equal to 'a', t is equal to minus of b minus 'a' divided by two and when x is equal to b, t is equal to b minus 'a' divided by two.

Therefore mean deviation
Is equal to 0ne divided by b minus 'a' into integral from minus of b minus 'a' divided by two to b minus 'a' divided by two modulus of t d t
Since it is an even function, we can change the limits as two times zero to b minus 'a' divided by two. Hence we get
two divided by b minus 'a' into integral from zero to b minus 'a' divided by two t d t
Is equal to two divided by b minus 'a' into t square divided by two, ranges from zero to b minus 'a' divided by two.
Is equal to b minus 'a' divided by four.

4. Results

Now consider the following result.
Show that for a rectangular distribution with probability density function f of (x) is equal to 0ne divided by two into ‘a’, where (minus a less than x less than a), moment generating function about origin is (sin h ‘a’ into t)/’a’ into t. Also show that moments of even order are given by, mu two n is equal to a power two into n divided by two into n plus 0ne.

Let us prove this result as follows.
Moment generating function about origin is given by,
M X of t is equal to Expectation of e to the power t into X
Is equal to integral from minus ‘a’ to ‘a’ e to the power t into x into f of x d x
Is equal to 0ne divided by two into ‘a’ into integral from minus ‘a’ to ‘a’, e to the power t into x d x
On integrating the function we get,
0ne divided by two into ‘a’ into e to the power t into x divided by t, ranges from minus ‘a’ to ‘a’
Is equal to 0ne divided by two into ‘a’ into e to the power minus ‘a’ into t minus e to the power ‘a’ into t
Is equal to sin h ‘a’ into t divided by two into ‘a’

Now expanding for sin h ‘a’ into t we get
0ne divided by ‘a’ into t into ‘a’ into t plus ‘a’ into t the whole cube divided by three factorial plus ‘a’ into t the whole power five divided by five factorial plus etc. till , plus ‘a’ into t whole power two into n plus divided by two into n plus 0ne factorial plus etc. an so on

This is equal to 0ne plus ‘a’ into t the whole square divided by three factorial plus ‘a’ into t the whole power four divided by five factorial plus etc. till, plus ‘a’ into t the whole power two into n divided by two into n plus one factorial plus etc. and so on

Since there are no terms with odd powers of t in M X of(t), all moments of odd order about origin vanish, that is mu dash two into n plus 0ne (about origin) is equal to zero
In particular mu 0ne dash (about origin) is equal to zero, that is Mean is equal to zero.
Thus mu r dash (about origin) is equal to mu r.
Hence mu two n is equal to coefficient of t to the power two into n divided by (two into n) factorial is equal to ‘a’ to the power two into n divided by (two into n plus 0ne).

On the x axis, (n plus 0ne) points are taken independently between the origin and x is equal to 0ne, all positions being equally likely. Show that the probability that the (k plus 0ne)th of this point, counted from origin lies in the interval x minus half d x to x plus half d x is
n c k into n plus 0ne into x to the power k into 0ne minus x to the power n minus k d x
Verify that integral of this expression from x is equal to zero to x is equal to 0ne is unity

Let us prove this result as follows.
Here X is given to be a random variable, uniformly distributed on [zero and 0ne].
Therefore f of (x) is equal to 0ne, zero less than or equal to x less than or equal to one
Now consider,
Probability of zero less than X less than x is equal to integral from zero to x f of x d x

Is equal to integral from zero to x one d x is equal to x. Name it as (one)
Therefore, Probability of (X greater than x) is equal to one minus Probability of (X less than or equal to x) is equal to one minus x. Name it as (two)

Also
Probability of x minus d x divided by two is less than X is less than x plus d x divided by two
Is equal to integral from x minus d x divided by two to x plus d x divided by two, f of x d x
Is equal to d x.
Required probability ‘p’ is given by
p is equal to Probability that [out of (n plus one) points, k point lies in closed interval [zero, and x minus (d x divided by two)] and out of the remaining (n plus one minus k) points, (n minus k) points lie in (x plus d x divided by two and one) and one point lies in (x minus d x divided by two and x plus d x divided by two). Name it as (three)

On using (one), (two) and (three) respectively we get,
p is equal to n plus one c k into x power k, into n plus one minus k c n minus k into one minus x the whole to the power n minus k d x.

Therefore p is equal to n plus one factorial divided by k factorial into n plus one minus k factorial, into x to the power k, into n plus one minus k factorial divided by n minus k factorial, into one minus x the whole to the power n minus k d x.
This is equal to n c k into n plus one into x power k into one minus x whole power n minus k d x

Now let us do the verification.
Consider the above function as I
Hence I is equal to integral from zero to one, n c k into n plus one into x to the power k into one minus x the whole to the power n minus k d x.
To simplify this we use the beta integral (which we study in detail in the coming modules)
Integral from zero one x power m minus one into one minus x to the power n minus one dx is equal to B of (m, n,)
This is equal to gamma m into gamma n divided by gamma m plus n where m is greater than zero and n is greater than zero.

Hence we can write,
I is equal to n c k into n plus one into B of k plus one, n minus k plus one
Implies, I is equal to n c k into n plus into gamma k plus one into gamma n minus k plus one divided by gamma n plus two
Using the relation gamma n is equal to n minus one factorial we get,
I is equal to n factorial divided by k factorial into n minus k factorial, into n plus one, into k factorial into n minus k factorial divided by n plus one factorial.
On simplification we get one.

5. Exercise

Exercise 1
If X is uniformly distributed with mean 1 and variance four by three, find P(X<0).
Let us solve the above as follows.
Let X follow Uniform (a,b) so that f of (x) is equal to one divided by (b minus a), where a less than x less than b. We have,
Mean (b plus 'a' divided by two) is equal to one
Implies, b plus 'a' is equal to two
Variance (b minus 'a') the whole square divided by twelve is equal to four divided by three
Implies (b minus 'a') the whole square is equal to sixteen or (b minus a) is equal to plus or minus four.

On solving by taking b minus 'a' is equal to minus four, we get 'a' is equal to minus one and b is equal to three
Or by taking b minus 'a' is equal to four, we get 'a' is equal to three and b is equal to minus one.
But in uniform distribution, we should have, a is less than b, the solution for 'a' and b is,
A is equal to minus one and b is equal to three
Therefore f of (x) is equal to one divided by four, where minus one less than x less than three.
Hence we can find
Probability that X less than zero is equal to integral from minus one to zero f of x d x
Is equal to one divided by four into x, ranges from minus one to zero
Is equal to one divided by four.

Subway trains on a certain line run every half hour between mid-night and six in the morning. What is the probability that a man entering the station at a random time during this period will have to wait at least twenty minutes?

Solution: Let the random variable X denotes the waiting time (in minutes) for the next train. Under the assumption that a man arrives at the station at random, X is distributed uniformly on (zero, thirty) with probability density function f of (x) is equal to one divided by thirty; where 0 less than x less than thirty.

The probability that he has to wait at least twenty minutes is given by,
Probability that X greater than or equal to twenty is equal to integral from twenty to thirty f of x d x is equal to one divided by thirty into integral from twenty to thirty, one into d x
Is equal to one divided by thirty into x, ranges from twenty to thirty which is equal to 0ne divided by three.

Here's a summary of our learning in this session where we have :

- Understood the definition of uniform distribution through pdf
- Described Distribution function
- Explained Moments that is, mean and variance
- Understood Moment Generating Function
- Understood Median and
- Explained Mean deviation