Frequently Asked Questions

1. Define uniform distribution on interval (a, b)

   Answer:
   A continuous random variable X is said to follow uniform distribution in the interval (a, b) if its probability density function is given by,
   $f(x) = 1/(b-a),\ 0<x<1,$

2. Write the pdf of uniform distribution on ((-θ, θ)

   Answer:
   $f(x) = 1/2\theta,\ -\theta<x<\theta,$

3. Why Uniform distribution is known as rectangular distribution?

   Answer:

   The distribution is also known as rectangular distribution, since the curve y=f(x) describes a rectangle over the x-axis and between the ordinates x=a and x=b

4. Obtain cumulative distribution function of the uniform distribution on (a, b)

   Answer:
   The cumulative distribution function of the uniform distribution on (a, b) is given by,

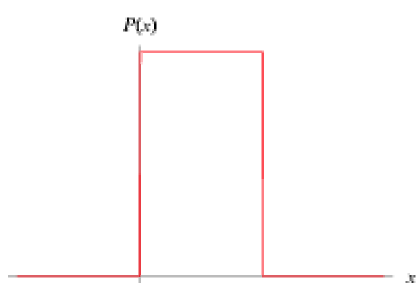
   $$F(X) = \int_a^x f(x)dx = \frac{1}{b-a}\int_a^x dx = \frac{1}{b-a}\left|X\right|_a^x = \frac{x-a}{b-a}$$

   Hence we can write,

   $$F(x) = \begin{cases} 0 & , \quad x \le a \\ (x-a)/(b-a) & , \quad a<x<b \\ 1 & , \quad x \ge b \end{cases}$$

5. Sketch the curve of a uniform distribution on (a, b)?

   Answer:



a b

6. Obtain mean of the distribution.

   Answer:
   If X~U (a, b), then the mean of the distribution is given by

$$E(X) = \int_x x.f(x)dx = \frac{1}{b-a}\int_a^b x dx = \frac{1}{b-a}\left|\frac{x^2}{2}\right|_a^b$$

$$= \frac{1}{2(b-a)}(b^2 - a^2) = \frac{(b+a)(b-a)}{2(b-a)} = \frac{(b+a)}{2}$$

7. Obtain variance of the distribution.

   Answer:
   If X~U (a, b) the variance of the distribution is given by, $V(x) = E(x^2) - [E(x)]^2$
   First let us find $E(x^2)$

$$E(X^2) = \int_x x^2.f(x)dx = \frac{1}{b-a}\int_a^b x^2 dx = \frac{1}{b-a}\left|\frac{x^3}{3}\right|_a^b$$

$$= \frac{1}{3(b-a)}(b^3 - a^3) = \frac{(b^2 + ab + b^2)(b-a)}{3(b-a)} = \frac{(b^2 + ab + b^2)}{3}$$

   Also

$$E(X) = \int_x x.f(x)dx = \frac{1}{b-a}\int_a^b x dx = \frac{1}{b-a}\left|\frac{x^2}{2}\right|_a^b$$

$$= \frac{1}{2(b-a)}(b^2 - a^2) = \frac{(b+a)(b-a)}{2(b-a)} = \frac{(b+a)}{2}$$

   Hence

$$V(X) = E(X^2) - [E(X)]^2 = \frac{b^2 + ab + a^2}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{4(b^2 + ab + a^2) - 3(b+a)^2}{12} = \frac{(b-a)^2}{12}$$

8. Write an expression for finding r^{th} raw moment.

   Answer:
   In general, moments of the uniform distribution with parameters a and b can be obtained as follows:

$$\mu_r' = E(X^r) = \int_x x^r . f(x)dx = \frac{1}{b-a}\int_a^b x^r dx = \frac{1}{b-a}\left|\frac{x^r}{r+1}\right|_a^b = \frac{1}{b-a}\left(\frac{b^r - a^r}{r+1}\right)$$

9. Obtain an expression for mgf of the distribution.

Answer:

If X~U (a, b) then Moment generating function of uniform distribution can be obtained as follows

$$E(e^{tX}) = \int_x e^{tx} f(x)dx = \frac{1}{b-a}\int_a^b e^{tx} dx = \frac{1}{b-a}\left|\frac{e^{tx}}{t}\right|_a^b = \frac{1}{t(b-a)}\left(e^{tb} - e^{ta}\right) \text{ at } t \neq 0$$

10. Find median of the distribution.

Answer:

If X~U (a, b) and M is the median of the distribution then, $\int_a^M f(x)dx = \frac{1}{2} = \int_M^b f(x)dx$

Hence we get M by solving any one of the integral. Hence consider,

$$\int_a^M f(x)dx = \frac{1}{2}$$

$$\Rightarrow \frac{1}{b-a}\int_a^M dx = \frac{1}{2}$$

$$\Rightarrow \frac{1}{b-a}|X|_a^M = \frac{1}{2}$$

$$\Rightarrow \frac{1}{b-a}(M-a) = \frac{1}{2}$$

On simplification, we get, M= (b+a)/2

11. Obtain mean deviation about mean.

Answer:

If X~U (a, b) then Mean deviation about mean is given by,

E|X-E(X)|=E|X-(b+a)/2|= $\int_x \left|x - \frac{b+a}{2}\right| . f(x)dx$

Let us take, $t = x - \frac{b+a}{2}$ so that when x=a, t=-(b-a)/2 and x=b, t= (b-a)/2

$$\therefore MD = \frac{1}{b-a}\int_{-(b-a)/2}^{(b-a)/2} |t| dt = \frac{2}{b-a}\int_0^{(b-a)/2} t dt = \frac{2}{b-a}\left|\frac{t^2}{2}\right|_0^{(b-a)/2} = \frac{b-a}{4}$$

12. Show that for a rectangular distribution with pdf. f(x) =1/2a, -a<x<a, mgf about origin is (sin h at)/at. Also show that moments of even order are given by, $\mu_{2n} = a^{2n}/(2n+1)$

Answer:
Mgf about origin is given by,

$$M_X(t) = E(e^{tX}) = \int_{-a}^{a} e^{tx} f(x)dx = \frac{1}{2a}\int_{-a}^{a} e^{tx}dx$$

$$= \frac{1}{2a}\left|\frac{e^{tx}}{t}\right|_{-a}^{a} = \frac{1}{2a}(e^{-at} - e^{at}) = \frac{\sinh(at)}{2a}$$

$$= \frac{1}{at}\left\{at + \frac{(at)^3}{3!} + \frac{(at)^5}{5!} + \ldots + \frac{(at)^{2n+1}}{(2n+1)!} + \ldots\right\}$$

$$= 1 + \frac{(at)^2}{3!} + \frac{(at)^4}{5!} + \ldots + \frac{(at)^{2n}}{(2n+1)!} + \ldots$$

Since there are no terms with odd powers of t in $M_X(t)$, all moments of odd order about origin vanish, i.e., μ'_{2n+1}(about origin)=0
In particular μ_1'(about origin)=0, i.e., Mean=0
Thus μ_r'(about origin)=μ_r
Hence μ_{2n} =coefficient of $t^{2n}/(2n)! = a^{2n}/(2n+1)$

13. On the x-axis, (n+1) points are take independently between the origin and x=1, all positions being equally likely. Show that probability that the $(k+1)^{th}$ of these points, counted from origin lies in the interval x-½dx to x+½dx is $\binom{n}{k}(n+1)x^k(1-x)^{n-k}dx$

Answer:
Here X is given to be a random variable, uniformly distributed on [0,1].
Therefore f(x)=1, 0≤x≤1

Now consider, $P(0 < X < x) = \int_0^x f(x)dx = \int_0^x 1.dx = x$

Therefore, P(X>x)=1-P(X≤x)=1-x -----------(2)

Also $P(x - \frac{dx}{2} < X < x + \frac{dx}{2}) = \int_{x-\frac{dx}{2}}^{x+\frac{dx}{2}} f(x)dx = dx$

Required probability 'p' is given by
p=P[out of (n+1) points, k point lie in closed interval [0, x-(dx/2)] and out of the remaining (n+1-k) points, (n-k) points lie in $\left(x + \frac{dx}{2}, 1\right)$ and one point lies in $\left(x - \frac{dx}{2}, x + \frac{dx}{2}\right)$ ---(3)

On using (1), (2) and (3) respectively we get,

$$p = \left[\binom{n+1}{k}x^k\right] \times \left[\binom{n+1-k}{n-k}(1-x)^{n-k}\right]dx$$

$$\therefore p = \frac{(n+1)!}{k!(n+1-k)!}x^k \frac{(n+1-k)!}{(n-k)!}(1-x)^{n-k}dx = \binom{n}{k}(n+1)x^k(1-x)^{n-k}dx$$

14. If X is uniformly distributed with mean and variance 4/3, find P(X<0).

Answer:

Let X~U(a,b) so that f(x)=1/(b-a), a<x<b
Mean (b+a)/2=1 Implies, b+a=2
Variance $=(b-a)^2/12=4/3$ Implies $(b-a)^2=16$ or (b-a)=±4
On solving we get, a=-1 and b=3 or a=3 and b=-1
But in uniform distribution, we should have, a<b, the solution for a and b is,
a=-1 and b=3. Therefore f(x)=¼;-1<x<3

Hence we can find $P(X < 0) = \int_{-1}^{0} f(x)dx = \frac{1}{4}\left|X\right|_{-1}^{0} = \frac{1}{4}$

15. Subway trains on a certain line run every half hour between mid-night and six in the morning. What is the probability that a man entering the station at a random time during this period will have to wait at least twenty minutes?

Answer:
Let the random variable X denotes the waiting time (in minutes) for the next train. Under the assumption that a man arrives at the station at random, X is distributed uniformly on (0, 30) with pdf f(x) =1/30; 0<x<30.
The probability that he has to wait at least 20 minutes is given by,

$$P(X \geq 20) = \int_{20}^{30} f(x)dx = \frac{1}{30}\int_{20}^{30} 1.dx = \frac{1}{30}\left|X\right|_{20}^{30} = \frac{1}{3}$$