1. Introduction

Welcome to the series of E-learning modules on Bivariate Moments and definition of Raw and Central Product Moments.

By the end of this session, you will be able to:

- Explain the bivariate moments
- Explain the bivariate raw moments
- Explain the bivariate central moments
- Explain the variance of sum and difference of two variables

The mathematical expectation of a function g of (x, y) of two dimensional variable (X, y) with probability density function f of (x, y) is given by
Expectation of g of X, Y is equal to double integral from minus infinity to infinity g of x, y into f of x, y dx dy (If X and Y are continuous variables)
Expectation of g of X Y is equal to summation over i, summation over j, xi, into yj into probability of X is equal to xi intersection Y is equal to yj. (If X and Y are discrete variables) provided the expectations exists.

2. Bivariate Raw & Central Moment

Now, let us define bivariate raw moment.
In particular, the rth and sth product moment about origin of the random variables X and Y is defined as
Mu r s dash is equal to double integral from minus infinity to infinity x power r into y power s into f of x, y dx dy
Or mu r s dash is equal to summation over i, summation over j, xi power r into yj power s into probability of X is equal to xi, intersection Y is equal to yj.
Hence, bivariate joint moments of X Y are
Expectation of X, Expectation of Y, are first order raw moments.
Expectation of X square, Expectation of X into Y and Expectation of Y square are 2^{nd} order raw moments.
Expectation of X cube, Expectation of X square into Y, Expectation of X into Y square and Expectation of Y cube are 3^{rd} order raw moments.

Now, let us define bivariate central moment.
The joint rth central moment of X and sth central moment of Y is given by
Mu r, s is equal to expectation of X minus expectation of X whole power r into Y minus expectation of Y whole power s
Is equal to expectation of X minus mu x whole power r into Y minus mu y whole power s, where Expectation of (X) is equal to mu x, Expectation of (Y) is equal to mu y
Hence, bivariate central moments of X Y are
Expectation of X minus E of X and Expectation of Y minus E of Y are first order central moments.
Expectation of X minus E of X the whole square, Expectation of X minus E of X into Y minus E of Y and Expectation of Y minus E of Y whole square are 2^{nd} order central moments.
Expectation of X minus E of X the whole cube, Expectation of X minus E of X the whole square into Y minus E of Y, Expectation of X minus E of X into Y minus E of Y the whole square and Expectation of Y minus E of Y the whole cube are third order central moments.

3. Note on Variance & Covariance

In particular, mu double zero dash is equal to 1 is equal to mu double zero, mu 1 not is equal to zero is equal to mu not 1
Mu 1 not dash is equal to Expectation of (X), mu not 1 dash is equal to expectation of (Y), mu 2 not is equal to variance of X is equal to sigma x square, mu not 2 is equal to Variance of (Y) is equal to sigma y square and mu 1, 1 is equal to Covariance of (X, Y)

We can also write, the central moments using the raw moments, in particular variance and covariance as follows:
Variance of (X) is equal to Expectation of [X minus Expectation of the whole square,
Is equal to Expectation of [X square plus {Expectation of (X) the whole square} minus 2 into X into Expectation of (X)]
Is equal to Expectation of (X square) plus {Expectation of (X) the whole square} minus 2 into Expectation of (X) into expectation of (X)
Is equal to Expectation of (X square) minus {Expectation of (X) the whole square}
Similarly, Variance of (Y) is equal to Expectation of (Y square) minus {Expectation of (Y) square}
Now, consider Covariance
Covariance of (X,Y) is equal to Expectation of [{(X minus Expectation of (X)} into {Y minus Expectation of (Y)}]
Is equal to Expectation of [X into Y minus Y into Expectation of (X) minus X into Expectation of (Y) plus Expectation of (X) into Expectation of (Y)]
Is equal to Expectation of (X into Y) minus Expectation of (Y) into Expectation of (X) minus Expectation of (X) into Expectation of (Y) plus Expectation of (X) into Expectation of (Y)
Is equal to Expectation of (X into Y) minus Expectation of (X) into Expectation of (Y)

Consider the following result.
For any two random variables, X and Y,
Variance of (a into X plus or minus b into Y) is equal to a square into Variance of (X) plus b square into Variance of (Y) plus or minus 2 into a into b into Covariance of (X, Y).
Let us prove the result as follows.
Consider
Variance of (a into X plus or minus b into Y)
Is equal to Expectation of [(a into X plus or minus b into Y) minus Expectation of (a into X plus or minus b into Y)] whole square.
Is equal to Expectation of [(a into X plus or minus b into Y) minus {a into Expectation of (X) plus or minus b into Expectation of (Y)}] the whole square.
Is equal to Expectation of [a into {X minus Expectation of (X)} plus or minus b into {Y minus Expectation of (Y)}] the whole square
On squaring and taking expectation to each of the terms, we get
a square into Expectation of [X minus Expectation of (X)] the whole square plus b square into Expectation of [Y minus Expectation of (Y)] the whole square plus or minus 2 into a into b into expectation of X minus expectation of X into expectation of Y minus expectation of Y
Is equal to a square into Variance of (X) plus b square into Variance of (Y) plus or minus 2 into a into b into Covariance of (X, Y)

4. Illustrations - 1

Illustration - 1
Consider an illustration on continuous data.
Two random variables X and Y have the following joint probability density function.
f of (x,y) is equal to k into (4 minus x minus y); where zero less than or equal to x less than or equal to 2, and zero less than or equal to y less than or equal to 2.
Find constant k, Expectation of (X), Expectation of (Y), Variance of (X), Variance of (Y) and Covariance of (X, Y). Also, write Expectation of X plus Y and Expectation of X minus Y.

Solution:
We can solve the above problem as follows.
Since f of x, y is a probability density function, double integral, f of x, y dx dy is equal to 1.
That is, k into double integral from zero to 2, 4 minus x minus y dx dy is equal to 1
Integrating over x and simplifying we get,
K into integral from zero to 2 6 minus 2 into y dy is equal to 1
Integrating over x and simplifying we get,
K into 8 is equal to 1 implies, k is equal to 1 divided by 8.

Before we find Expectation and variance, we find marginal distribution of X and Y.
Marginal distribution of X is given by,
f of x is equal to 1 divided by 8 into integral from zero to 2, 4 minus x minus y dy
On integrating the function and simplifying we get,
1 divided by 8 into 6 minus 2 x
By taking 2 common outside the bracket and cancelling it with the denominator we get, 1 divided by 4 into 3 minus x, where zero less than or equal to x less than or equal to 2.
Marginal distribution of Y is given by,
f of y is equal to 1 divided by 8 into integral from zero to 2 4 minus x minus y dx
On integrating the function and simplifying we get,
1 divided by 8 into 6 minus 2 y
By taking 2 common outside the bracket and cancelling it with the denominator we get, 1 divided by 4 into 3 minus y, where zero less than or equal to y less than or equal to 2.
Now, let us obtain Expectation.
Expectation of X is equal to integral from zero to 2 x into f of x d x
Is equal to 1 divided by 4 into integral from zero to 2, x into 3 minus x dx
On integration and simplifying we get,
5 divided by 6.
Expectation of Y is equal to integral from zero to 2 y into f of y dy
Is equal to 1 divided by 4 into integral from zero to 2, y into 3 minus y dy
On integration and simplifying we get,
5 divided by 6.

To find variance, first we find expectation of X square and expectation of Y square.
Expectation of X square is equal to integral from zero to 2 x square into f of x dx
Is equal to 1 divided by 4 into integral from zero to 2, x square into 3 minus x dx is equal to 1
Similarly, expectation of Y square is equal to integral from zero to 2, y square into f of y dy
Is equal to 1 divided by 4 into integral from zero to 2 y square into 3 minus y dy is equal to 1

Variances are given as follows:
Variance of (X) is equal to Expectation of (X square) minus {Expectation of (X)} the whole square is equal to 1 minus (5 divided by 6) whole square is equal to 11 divided by 36.
Variance of (Y) is equal to Expectation of (Y square) minus {Expectation of (Y)} the whole square is equal to 1 minus (5 divided by 6) whole square is equal to 11 divided by 36.
We know that Covariance of (X,Y) is equal to Expectation of (X into Y) minus Expectation of (X) into Expectation of (Y)
We have already found Expectation of (X) and Expectation of (Y).

Now, let us obtain Expectation of (X into Y).
Expectation of X into Y is equal to double integral from zero to 2, x into y into f of x, y dx dy.
Is equal to 1 divided by 8 into integral from zero to 2, y into integral from zero to 2, x into 4 minus x minus y dx dy
By integrating the terms in the bracket and simplifying we get,
1 divided by 8 into integral from zero to 2, y into 16 divided by 3 minus 2 into y dy
On integrating the function with respect to y and then simplifying we get, 2 divided by 3.
Hence, covariance of X Y is equal to 2 divided by 3 minus 5 divided by 6 into t divided by 6
On simplifying the equation, we get, minus 1 divided by 36.
We know that for any 2 variables, Expectation of X plus Y is equal to Expectation of X plus expectation of Y
Is equal to 5 divided by 6 plus 5 divided by 6 is equal to 10 divided by 6
Also, expectation of X minus Y is equal to expectation of X minus Expectation of Y
Is equal to 5 divided by 6 minus 5 divided by 6 is equal to zero.

5. Illustration – 2

Illustration – 2
Consider an illustration on discrete data.
X and Y have a bivariate distribution given by
Probability of X is equal to x intersection Y is equal to y is equal to x plus 3 into y whole divided by 24; where x, y take values, 1, 1; 1, 2; 2, 1 and 2, 2
Find Expectation of (X), Expectation of (Y), Variance of (X), Variance of (Y) and Covariance of (X,Y).

Solution:
First, let us obtain marginal distribution of X and Y.
Marginal distribution of X is given by,
Probability of X is equal to x is equal to summation over y is equal to 1 to 2 x plus 3 into y whole divided by 24
Is equal to x plus 3 into 1 whole divided by 24 plus x plus 3 into 2 whole divided by 24
Is equal to 2 into x plus 9 whole divided by 24, where x is equal to 1 and 2.
Marginal distribution of Y is given by,
Probability of Y is equal to y is equal to summation over x is equal to 1 to 2, x plus 3 into y whole divided by 24
Is equal to 1 plus 3 into y whole divided by 24 plus 2 plus 3 into y whole divided by 24
Is equal to 3 plus 6 into y divided by 24, where y is equal to 1 and 2.

Now, let us find the expectations.
Expectation of X is equal to summation over x is equal to 1 to 2 x into Probability of X is equal to x
Is equal to summation over x is equal to 1 to 2, x into 2 into x plus 9 divided by 24.
Is equal to 1 into 2 into 1 plus 9 divided by 24 plus 2 into 2 into 2 plus 9 divided by 24
Is equal to 1 into 11 by 24 plus 2 into 13 by 24 is equal to 37 by 24.
Expectation of Y is equal to summation over y is equal to 1 to 2, y into probability of Y is equal to y
Is equal to summation over y is equal to 1 to 2, y into 3 plus 6 into y whole divided by 24.
Is equal to 1 into 3 plus 6 into 1 whole divided by 24 plus 2 into 3 plus 6 into 2 whole divided by 24.
Is equal to 1 into 9 by 24 plus 2 into 15 by 24 is equal to 39 by 24 is equal to 13 by 8.

Before we find the variance, let us find expectation of X square and Expectation of Y square.
Expectation of X square is equal to summation over x is equal to 1 to 2, x square into Probability of X is equal to x
Is equal to summation over x is equal to 1 to 2, x square into 2 into x plus 9 divided by 24
Is equal to 1 square into 2 into 1 plus 9 divided by 24 plus 2 square into 2 into 2 plus 9 divided by 24
Is equal to 1 into 11 by 24 plus 4 into 13 by 24 is equal to 63 by 24.
Is equal to 21 divided by 8.

Expectation of Y square is equal to summation over y is equal to 1 to 2, y square into probability of Y is equal to y

Is equal to summation over y is equal to 1 to 2, y square into 3 plus 6 into y whole divided by 24
Is equal to 1 square into 3 plus 6 into 1 whole divided by 24 plus 2 square into 3 plus 6 into 2 whole divided by 24
Is equal to 1 into 3 plus 6 into 1 whole divided by 24 plus 4 into 3 plus 6 into 2 whole divided by 24
Is equal to 69 divided by 24 is equal to 23 divided by 8.
Variance of X is equal to expectation of X square minus Expectation of X the whole square
Is equal to 21 by 8 minus 37 by 24 whole square
Is equal to zero point 2, 4, 8, 3.
Variance of Y is equal to expectation of Y square minus expectation of Y the whole square
Is equal to 23 by 8 minus 13 by 8 the whole square
Is equal to zero point 2, 3 4, 4.
To find covariance of x, y, let us find expectation of X into Y
Expectation of X Y is equal to summation over x is equal to 1 to 2, summation over y is equal to 1 to 2 x into y into probability of X is equal to x intersection Y is equal to Y
Is equal to summation over x is equal to 1 to 2, summation over y is equal to 1 to 2 x into y into x plus 3 into y divided by 24
Is equal to summation over x is equal to 1 to 2, x into summation over y is equal to 1 to 2, y into x plus 3 into y whole divided by 24
Is equal to summation over x is equal to 1 to 2, x into 1 into x plus 3 into 1 whole divided by 24 plus 2 into x plus 3 into 2 whole divided by 24
Is equal to summation over x is equal to 1 to 2, x into 3 into x plus 15 whole divided by 24
Is equal to 1 into 3 into 1 plus 15 whole divided by 24 plus 2 into 3 into 2 plus 15 whole divided by 24
Is equal to 60 divided by 24
Is equal to 5 by 2.

Hence, covariance is given by,
Covariance of X Y is equal to expectation of X into Y minus Expectation of X into Expectation of Y
Is equal to 5 divided by 2 minus 37 divided by 24 into 13 divided by 8
Is equal to zero point zero zero 5, 2.

Here's a summary of our learning in this session, where we have understood:

- The bivariate moments
- The bivariate raw moments
- The bivariate central moments
- The variance of sum and difference of two variables