1. Introduction

Welcome to the series of E-learning modules on Bivariate Discrete and Continuous Distribution, its probability mass function (pmf) and probability density function (pdf).

By the end of this session, you will be able to:

- Explain the two dimensional random variable – Discrete and Continuous
- Explain the joint probability mass function
- Explain the distribution function
- Explain the probability density function
- Explain the conditions for a probability function to be a pmf/pdf

Before we start discussion about bivariate distribution and their probability functions, let us define two dimensional random variables.
Suppose we are interested in recording the height and weight of every person in a certain educational institution, then to describe such experiments mathematically, we introduce the study of two random variables.

2. Two Dimensional Random Variables

Let us define a two dimensional random variables.
Let X and Y be two random variables defined on the same sample space S, then the function (X, Y) that assigns a point R square (is equal to R into R) is called a two dimensional random variable.
Let (X, Y) be a two dimensional random variable defined on the sample space S and omega belongs to S. The value of (X, Y) at omega is given by the pair of real numbers {X of omega, Y of omega}. The notation {X less than or equal to a, Y less than or equal to b} denotes the event of all elements omega belongs to S, such that X of (omega) less than or equal to a, Y of (omega) less than or equal to b. The probability of the event {X less than or equal to a, Y less than or equal to b} will be denoted by P of (X less than or equal to a, Y less than or equal to b).

Let A is equal to {a less than X less than or equal to b}, B is equal to {c less than Y less than or equal to d}, be two events.
Then, the event
{a less than X less than or equal to b, c less than Y less than or equal to d} is equal to {a less than X less than or equal to b intersection c less than Y less than or equal to d} is equal to A intersection B
Therefore, Probability that (a less than X less than or equal to b intersection c less than Y less than or equal to d) is equal to Probability that (A intersection B)

Consider the following remarks:

- A two dimensional random variable is said to be discrete if it takes at most a countable number of points in R square and a two dimensional random variable is said to be continuous if it assumes all the values in R square.
- Two random variables X and Y are said to be jointly distributed if they are defined on the same probability space. The sample points consist of 2-tuples. If the joint probability function is denoted by P X Y of (x, y), then the probability of certain event E is given by,
  P X Y of (X, Y) is equal to Probability that [(X, Y) belongs to E]

3. Joint Probability Distribution Function and Verifications

Let us define probability mass function.
If (X, Y) is a two dimensional discrete random variable, then the joint discrete function of X, Y, also called joint probability mass function of X, Y denoted by p X, Y is defined as,
p X Y of (xi, yi) is equal to p of (X is equal to xi, Y is equal to yi) for a value (xi, yi) of (X, Y) and p X Y of (xi, yi) is equal to zero, otherwise.

Let us define joint probability distribution function.
The distribution of two dimensional random variable (X, Y) is a real valued function F defined for all real x and y by the relation F X Y of (x, Y) is equal to Probability that (X less than or equal to x, Y less than or equal to y)

Following are the properties of joint distribution function:
For the real numbers a1, b1, a2, and b2 probability that a1 less than X less than or equal to b1, a2 less than X less than or equal to b2
Is equal to F X Y of b1, b2 plus F X Y a1, a2 minus F X Y a1, b2 minus F X Y b1, a2.
Let a1 less than a2, b1 less than b2, then
X less than or equal to a1, Y less than or equal to a2 plus a1 less than X less than or equal to b1, Y less than or equal to a2 is equal to X less than or equal to b1, Y less than or equal to a2, and the events on Left Hand Side are mutually exclusive.

Therefore, F of (a_1, a_2) plus Probability of (a_1 less than X less than or equal to b_1, Y less than or equal to a_2) is equal to F of (b_1, a_2)
Implies, F of (b_1, a_2) minus F of (a_1, a_2) is equal to Probability of (a_1 less than X less than or equal to b_1, Y less than or equal to a_2)
Therefore, F of (b_1, a_2) greater than or equal to F of (a_1, a_2) [since Probability of (a_1 less than X less than or equal to b_1, Y less than or equal to a_2)is greater than or equal to zero)
Similarly, it follows that
F of (a_1, b_2) minus F of (a_1, a_2) is equal to Probability of (X is less than or equal to a_1, a_2 less than Y less than or equal to b_2)
Therefore, F of (a_1, b_2) is greater than or equal to F of (a_1, a_2), which shows that F of (x, y) is monotonic non decreasing function.
2. F of (minus infinity , y) is equal to zero is equal to F of (x, minus infinity), F of (plus infinity, plus infinity) is equal to 1
3. If the density function f of (x, y) is continuous at (x, y), then
2^{nd} order partial derivative, dou square F divided by dou x dou y is equal to f of x, y

From the joint distribution function, F X Y of (x, y) of bivariate continuous random variable, we get the joint probability density function by differentiation as follows:
f of x, y is equal to dou square F by dou x dou y
Is equal to limit delta x tends to zero, delta y tends to zero, Probability of x less than or equal to X less than or equal to x plus delta x, y less than or equal to Y less than or equal to y plus delta y divided by delta x delta y

Or it may be expressed in the following way also.
“The probability that the point (x, y) will lie in the infinitesimal rectangular region, of area dx dy is given by,
Probability that (x minus half dx less than or equal to X less than or equal to x plus dx, y minus half dy less than or equal to Y less than or equal to y plus half dy) is equal to d F X Y of (x, y)
and is denoted by f X Y of(x, y) dx dy, where f X Y of (x, y) is called the joint probability density function of X and Y.

Now, let us discuss the conditions necessary to verify whether the given probability function is pmf or pdf, or not.
To verify whether the given probability function is a probability mass function or probability density function or not, we check for following two conditions:

The first condition is,
P of (x, y) greater than or equal to zero, for all x, y (for discrete bivariate distribution)
f of (x, y) is greater than or equal to zero, for all x, y (for continuous bivariate distribution)

The second condition is,
Summation over X and Y, p of (x, y) is equal to 1 (for discrete bivariate distribution)
And integral over x and y, f of (x, y) dx dy is equal to 1 (for continuous bivariate distribution)

4. Illustration 1

Illustration 1:
Consider the following illustration on discrete distribution:
In the random placement of 3 balls in three cells, describe the possible outcomes of the experiment. Let X_i denote the number of balls in cell i, where i is equal to 1, 2, 3; and N be the number of cells occupied. Obtain the joint distribution of (X_1, N) and (X_1, X_2).

Solution:
a. Let the three balls be denoted as a, b and c. Then, the possible outcomes of placing the three balls in three cells are as follows:
Observe that, we have placed each ball in each of the three cells by changing their positions like, first we arrange the balls in 3 cells as a, b and c. In the next arrangement, first cell contains ball a, 2nd cell contains ball c and 3rd cell contains ball b and then we go on changing the position of balls in different cells. Once we finish all the arrangement by single ball per cell (from 1 to 6), we keep 2 balls in a cell, one ball in another cell and one cell is left blank (from 7 to 24) like, first cell contains balls a and b, 2nd cell contains ball b and the last cell is left blank. In the next arrangement, in first cell, balls a and b are kept, 2nd cell is left blank and ball c is kept in the 3rd cell. Likewise, we do all possible arrangements with 2 balls in a cell.
Finally, we keep all the 3 balls in a single cell and leave other 2 cells blank, that is, all the 3 balls are kept in first cell and other 2 cells are left blank. In the next arrangement, all the 3 balls are kept in 2nd cell and first and the last cells are left blank and in the last arrangement, first 2 cells are left blank and all the 3 balls are kept in the 3rd cell (from 25 to 27). Hence, all together we can have 27 arrangements of balls.
Each of these arrangements represents a sample event, that is, a sample point. The sample space contains 27 points.

Let N denote the number of occupied cells. The favorable cases for N is equal to 1 are at numbers 25, 26 and 27, that is, 3; for N is equal to 2 are at numbers 7 to 24, that is, 18 and for N is equal to 3 are at numbers 1 to 6, that is, 6. Accordingly, the probability distribution of N is
Probability of (N is equal to 1) is equal to 3 divided by 27
Probability of (N is equal to 2) is equal to 18 divided by 27 and
Probability of (N is equal to 3) is equal to 6 divided by 27.
Let X1 denote the number of balls placed in the first cell. Then, form the above table of sample point X1 can take values, 0, 1, 2 and 3.
Probability of (X_1 is equal to zero) is equal to 8 divided by 27;
Probability of (X_1 is equal to 1) is equal to 12 divided by 27;
Probability of (X_1 is equal to 2) is equal to 6 divided by 27 and
Probability of (X_1 is equal to 3) is equal to 1 divided by 27.

The joint distribution of N and X1 can be obtained as follows:
Probability of (N is equal to 1, X_1 is equal to 0) is equal to 2 divided by 27 (sample number 26 and 27)
Probability of (N is equal to 1, X_1 is equal to 1) is equal to 0; (as there is no cell in which only cell is occupied and has only 1 ball).
Similarly, we find all other probabilities.

Probability of (N is equal to 1, X_1 is equal to 2) is equal to 0;
Probability of (N is equal to 1, X_1 is equal to 3) is equal to 1 divided by 27;
Probability of (N is equal to 2, X_1 is equal to 0) is equal to 6 divided by 27;
Probability of (N is equal to 2, X_1 is equal to 1) is equal to 6 divided by 27;
Probability of (N is equal to 2, X_1 is equal to 2) is equal to 6 divided by 27;
Probability of (N is equal to 2, X_1 is equal to 3) is equal to 0;
Probability of (N is equal to 3, X_1 is equal to 0) is equal to 0;
Probability of (N is equal to 3, X_1 is equal to 1) is equal to 6 divided by 27;
Probability of (N is equal to 3, X_1 is equal to 2) is equal to 0;
Probability of (N is equal to 3, X_1 is equal to 3) is equal to 0;
The above probabilities are written in a tabular form to get joint distribution of N and X1 as follows.

Here is the table showing joint distribution of N and X_1.

Figure 1

X_1 \ N	1	2	3	Distribution of X_1
0	$^2/_{27}$	$^6/_{27}$	0	$^8/_{27}$
1	0	$^6/_{27}$	$^6/_{27}$	$^{12}/_{27}$
2	0	$^6/_{27}$	0	$^6/_{27}$
3	$^1/_{27}$	0	0	$^1/_{27}$
Distribution of N	$^3/_{27}$	$^{18}/_{27}$	$^6/_{27}$	1

Proceeding on the same lines, the joint distribution of X1 and X2 can be obtained as shown in the table below.

Figure 2

X_1 \ X_2	0	1	2	3	Distribution of X_1
0	$^1/_{27}$	$^3/_{27}$	$^3/_{27}$	$^1/_{27}$	$^8/_{27}$
1	$^3/_{27}$	$^6/_{27}$	$^3/_{27}$	0	$^{12}/_{27}$
2	$^3/_{27}$	$^3/_{27}$	0	0	$^6/_{27}$
3	$^1/_{27}$	0	0	0	$^1/_{27}$
Distribution of X_2	$^8/_{27}$	$^{12}/_{27}$	$^6/_{27}$	$^1/_{27}$	1

Observe that both x1 and X2 are number of balls per cell, it take values, zero, 1, 2 and 3. The first cell entry is both first and 2nd cells are empty. That is, sample number 27. Hence, the probability is 1 divided by 27.

The second entry in the first row is that first cell does not contain any ball and 2nd cell has one ball, that is sample number 18, 21 and 24, that is three cases. Hence, the probability is 3 divided by 27. Similarly, we write all the probabilities to fill the table accordingly.

5. Illustrations 2 and 3

Illustration – 2
For the following bivariate probability distribution of X and Y find, P(X is equal to 1,Y is equal to 1), P(X is less than or equal to 1,Y is equal to 2) and P(X less than 3, Y less than or equal to 4)

Figure 3

y / X	1	2	3	4	5	6
0	0	0	$1/32$	$2/32$	$2/32$	$3/32$
1	$1/16$	$1/16$	$1/8$	$1/8$	$1/8$	$1/8$
2	$1/32$	$1/32$	$1/64$	$1/64$	0	$2/64$

In the table, we have written the values taken by X that is 0, 1 and 2 one below the other and the values taken by y is 1, 2 up to 6 as column heading. The entries inside the table give the corresponding probabilities.
That is, probability of x is equal to zero and y is equal to 1 is zero,
Probability of x is equal to 1 and y is equal to 1 is 1 divided by 16 and so on.

Solution:
The probabilities can be found as follows:
Probability of (X is equal to 1 and Y is equal to 1) is the entry in 2^{nd} row and first column of the table, that is 1 divided by 16
P(X less than or equal to 1,Y is equal to 2)
The values taken by X is zero and 1 and is equal to
Probability of (X is equal to 0, Y is equal to 2) plus Probability of (X is equal to 1,Y is equal to 2) is equal to zero plus 1 divided by 16 is equal to 1 divided by 16
Probability of (X less than 3,Y less than or equal to 4)
Is equal to Probability of (X is equal to 0,Y is less than or equal to 4) plus Probability of (X is equal to 1,Y is less than or equal to 4) plus Probability of (X is equal to 2,Y less than or equal to 4)

Here, we consider the probability of Y corresponding to the values less than or equal to 4, that is zero 1, 2, 3 and 4.
Is equal to (zero plus zero plus 1 divided by 32 plus 2 divided by 32) plus (1 divided by 16 plus 1 divided by 16 plus 1 divided by 8 plus 1 divided by 8) plus (1 divided by 32 plus 1 divided by 32 plus 1 divided by 64 plus 1 divided by 64) is equal to 9 divided by 16

If X and Y are two random variables having joint density function f(x, y) is equal to 1 divided by 8 into (6 minus x minus y); where zero less than or equal to x less than 2, and 2 less than or equal to y less than 4. Find Probability of (X less than 1 intersection Y less than 3) and Probability of (X plus Y is less than 3)

Solution:
We can write,
Probability of (X less than 1 intersection Y less than 3) is equal to double integral from minus infinity to 1 and minus infinity to 3 f of x dx dy
Is equal to 1 divided by 8 into double integral from zero to 1 and 2 to 3, 6 minus x minus y dx dy
Is equal to 3 divided by 8.

To find Probability of (X plus Y less than 3), let us find the limits for X and Y. X plus Y is less than 3 implies Y is less than 3 minus X.
The minimum value taken by Y is equal to 2. Hence, X cannot be greater than 1.
Therefore,
Probability of X plus Y less than 3 is equal to double integral from zero to 1 and 2 to 3 minus x f of x dx dy
Is equal to 1 divided by 8 into double integral from zero to 1 and 2 to 3 minus x 6 minus x minus y dx dy
Is equal to 5 divided by 24

Here's a summary of our learning in this session, where we have understood:

- The two dimensional random variable – Discrete and Continuous
- The joint probability mass function
- The distribution function
- The probability density function
- The conditions for a probability function to be a pmf/pdf