1. Introduction

Welcome to the series of E-learning modules on Laplace Distribution.

By the end of this session, you will be able to:

- Explain Standard Laplace Distribution
  - Moments, skewness and kurtosis
  - mgf and cgf
- Explain Laplace distribution from exponential distribution
- Explain Laplace distribution with two parameters
  - Moments, mgf, and cgf
  - Skewness and kurtosis
- Explain uses and applications of the distribution

In Probability Theory and Statistics, the Laplace Distribution is a continuous probability distribution named after Pierre-Simon Laplace. It is also called as the *double exponential distribution* because it can be thought of as two exponential distributions (with an additional location parameter) spliced together back-to-back.

This distribution is often referred to as Laplace's first law of errors. He published it in seventeen seventy four, when he noted that the frequency of an error could be expressed as an exponential function of its magnitude once its sign was disregarded.

2. Standard Laplace Distribution

Let us define standard Laplace distribution as follows:
A continuous random variable X is said to have Laplace (or double exponential) Distribution if its probability density function is given by,
f of x is equal to half into e power minus mod x; where minus infinity less than x less than infinity

If X is a standard Laplace variate, then rth raw moment is given by,
Mu r dash is equal to Expectation of X power r
Is equal to integral from minus infinity to infinity x power r into f of x dx
Is equal to half into x power r into e power minus mod x dx
Since we cannot say it as pure positive function or negative function, let us split integral from minus infinity to infinity as, minus infinity from zero and zero to infinity. Hence we get,
Half integral from minus infinity to zero x power r into e power minus mod x dx plus integral from zero to infinity, x power r into e power minus mod x dx
Is equal to half into minus 1 whole power r into integral from zero to infinity x power r into e power minus x dx plus x power r into e power minus x dx
Is equal to half into minus 1 power r into gamma r plus 1 plus gamma r plus 1
Taking the common term outside, we get,
Half into gamma r plus 1 into minus 1 power r plus 1
Is equal to half into r factorial into minus 1 power r plus 1

By substituting r is equal to 1, we get,
Mean mu 1 dash is equal to half into 1 factorial into minus 1 power 1 plus 1 is equal to zero.
Since mu 1 dash is equal to zero, all the moments about origin are same as the moments about the mean.
That is mu r dash is equal to mu r
Hence, when r is equal to 2, mu 2 dash is equal to mu 2 is equal to half into 2 factorial into [minus 1 square plus 1] is equal to 2
When r is equal to 3, mu 3 dash is equal to mu 3 is equal to half into 3 factorial into [minus 1 cube plus 1] is equal to zero and
When r is equal to 4, mu 4 dash is equal to mu 4 is equal to half into 4 factorial into [minus 1 power 4 plus 1] is equal to 24
Hence, we can find coefficient of skewness and kurtosis.
Hence, standard Laplace distribution has symmetric and leptokurtic curve.

Now, let us obtain the moment generating function.
If a random variable X is a standard Laplace variate, then moment generating function is given by
MX of t is equal to Expectation of (e power t into x)
Is equal to integral from minus infinity to infinity, e power t into x into f of x dx
Is equal to half into integral from minus infinity to infinity e power t into x into e power minus mod x dx
Now, let us split integral from minus infinity to infinity as minus infinity to zero and from zero to infinity.
Hence we can write,

Half into integral from minus infinity to zero e power t into x into e power minus mod x dx plus integral from zero to infinity e power t into x into e power minus mod x dx
Is equal to half integral from zero to infinity e power minus t into x into e power minus x dx plus integral from zero to infinity e power t into x into e power minus x dx
Is equal to half into integral from zero to infinity e power minus x into 1 plus t dx plus integral from zero to infinity e power minus x into 1 minus t dx

On integrating the two functions, we get,
MX of t is equal to half into 1 divided by 1 plus t plus 1 divided by 1 minus t
Is equal to 1 divided by 1 minus t square, which can also be written as 1 minus t square power minus 1.

Let us obtain Cumulant generating function of the distribution.
Cumulant generating function of standard Laplace distribution is given by,
KX of t is equal to log MX of t
Is equal to log (1 minus t square) whole power minus 1
Is equal to minus log (1 minus t square)
Is equal to minus of minus t square minus, minus t square the whole square by two plus, minus t square the whole cube by three minus etc.
Is equal to t square plus t power 4 divided by 2 plus t power 6 divided by 3 plus etc.

By equating the coefficients of t power r divided by r factorial, we get r^{th} cumulant.
Thus,
K1 is equal to coefficient of t is equal to zero, which is equal to mu 1 dash, that is mean.
K2 is equal to coefficient of t square divided by 2 factorial is equal to 2 which is equal to Variance
K3 is equal to coefficient of t cube divided by 3 factorial is equal to zero, which is equal to mu 3, and
K4 is equal to coefficient of t power 4 divided by 4 factorial is equal to 12
Hence, mu 4 is equal to K_4 plus 3 into K2 square is equal to 12 plus 3 into (2 square) is equal to 24.

3. Laplace Distribution from Exponential Distribution

We can obtain Laplace distribution as the distribution of difference of two independent exponential variates with parameter 1.
That is if X and Y are independent with a common probability density function f of (x) is equal to e power minus x, where x is greater than or equal to zero, then the distribution of X minus Y is standard Laplace variate.
We know that, moment generating function of exponential variate with parameter theta is given by, (1 minus t by theta) the whole power minus 1. When θ is equal to 1, moment generating function becomes (1 minus t) the whole power minus 1.

Since X and Y are identically distributed,
MX of (t) is equal to MY of (t) is equal to (1 minus t) whole power minus 1
M minus Y of (t) is equal to MY of (minus t) is equal to (1 plus t) whole power minus 1
Hence, MX minus Y of (t) is equal to MX plus (minus Y) of (t)
Is equal to MX of (t) into M minus Y of (t)
Is equal to (1 minus t) power minus 1 into (1 plus t) power minus 1
Is equal to (1 minus t square) the whole power minus 1, which is the moment generating function of standard Laplace distribution. Hence, by uniqueness theorem of moment generating function, X minus Y follows Standard Laplace Distribution.

Mean deviation about mean of standard Laplace distribution is given by
MD is equal to Expectation of mod X minus zero is equal to expectation of mod X
Is equal to integral from minus infinity to infinity mod x into f of x dx
Is equal to half into mod x into e power minus mod x dx. Since this is an even function, integral from minus infinity to infinity can be written as 2 times zero to infinity.
Therefore, we get, half into 2 into integral from zero to infinity x into e power minus x dx
X we can write as, x power 2 minus 1, so that we can make use of gamma function. Hence, we get, integral from zero to infinity x power 2 minus 1 into e power minus x dx, which is equal to gamma 2 is equal to 1.

A continuous random variable X is said to have a double exponential (Laplace) distribution with two parameters lambda and mu if its probability density function is given by,
f of x is equal to 1 divided by 2 into lambda into e power minus x minus mu divided by lambda, where minus infinity less than x less than infinity and lambda greater than zero.
We write X follows Laplace distribution with parameters (lambda and mu).
Let Y is equal to (x minus mu) divided by lambda implies X is equal to mu plus lambda into Y
The probability density function of Y is given by, g of y is equal to f of x into mod d x divided by d y is equal to 1 divided by 2 into lambda into e power minus mod y into lambda,
Is equal to half into e power minus mod y, minus infinity less than y less than infinity, which is the probability density function of standard Laplace Distribution. Hence, if X follows Laplace (lambda and mu), then Y follows Laplace (1 and zero)

4. Laplace Distribution with Two Parameters

Let us obtain the moments of Laplace distribution.
If X follows Laplace distribution with parameters (lambda and mu), the r^{th} moment about origin is given by,
Mu r dash is equal to Expectation of (X power r)
Is equal to integral from minus infinity to infinity x power r into f of x dx
Is equal to 1 divided 2 into lambda integral from minus infinity to infinity x power r into e power minus mod x minus mu divided by lambda dx
Is equal to half into integral from minus infinity to infinity, z into lambda plus mu the whole power r into e power minus z dz, where z is equal to x minus mu divided by lambda.
Using binomial expansion for z into lambda plus mu the whole power r, we get,
Half into integral from minus infinity to infinity, summation over k is equal to zero to r, r c k into z into lambda whole power k into mu power r minus k into e power minus mod z dz
Interchanging summation and integral and simplifying we get,
Half into summation over k is equal to zero to r, r c x into lambda power k into mu power r minus k into integral from minus infinity to infinity z power k into e power minus mod z dz

Is equal to half into summation over k is equal to zero to r, r c k into lambda power k into mu power r minus k into integral from minus infinity to zero z power k into e power minus mod z dz plus integral from zero to infinity, z power k into e power minus mod z dz
Is equal to half into summation over k is equal to zero to r, r c k into lambda power k into mu power r minus k into minus 1 power k into integral from zero to infinity z power k into e power minus z dz plus integral from zero to infinity z power k into e power minus z dz
Is equal to half into summation over k is equal to zero to r, r c k into lambda power k into mu power r minus k into gamma k plus 1 into minus 1 power k plus 1
Is equal to half into summation over k is equal to zero to r, r c k into lambda power k into mu power r minus k into k factorial into minus 1 power k plus
Hence, by substituting r is equal to 1 and 2 respectively we get, mean mu 1 dash is equal to mu and mu 2 dash is equal to mu square plus 2 into lambda square and hence variance mu 2 is equal to 2 into lambda square

Now, let us obtain moment generating function of the Laplace distribution with parameters lambda and mu.
MX of t is equal to Expectation of e power t into X
Is equal to integral from minus infinity to infinity e power t into x into f of x dx
Is equal to 1 divided by 2 into lambda into integral from minus infinity to infinity e power t into x into e power minus mod x minus mu divided by lambda dx.
Is equal to half into integral from minus infinity to infinity e power t into z into lambda plus mu into e power minus mod z where z is equal to x minus mu divided by lambda.
Is equal to e power t into mu divided by 2 into integral from minus infinity to zero e power t into z into lambda into e power minus mod z dz plus integral from zero to infinity e power t into z into lambda into e power minus mod z dz.
Is equal to e power t into mu divided by 2 into integral from zero to infinity e power t into

minus z into lambda into e power minus z dz plus integral from zero to infinity e power t into z into lambda into e power minus z dz.
Is equal to e power t into mu divided by 2 into integral from zero to infinity e power minus z into 1 plus t into lambda dz plus integral from zero to infinity e power minus z into 1 minus t into lambda dz

On integrating and simplifying two functions we get,
MX of t is equal to e power t into mu divided by 2 into 1 divided by 1 plus t into lambda plus 1 divided by 1 minus t into lambda.
On taking common denominator and simplifying, we get,
e power t into mu divided by 1 minus t into lambda the square
Is equal to e power t into mu into 1 minus t square lambda square whole power minus 1.

Let us obtain the Cumulant generating function of the Laplace distribution.
Cumulant generating function KX of t is equal to log MX of t
Is equal to log e power t into mu into 1 minus t square into lambda square whole power minus 1
Is equal to t into mu minus log 1 minus t square into lambda square
Is equal to t into mu minus, minus t square into lambda square minus, minus t square into lambda square the whole square divided by 2 plus, minus t square into lambda square the whole cube divided by 3 minus etc.
Is equal to t into mu plus t square into lambda square plus t power 4 into lambda power 4 divided by 2 plus t power 6 into lambda power 6 divided by 3 plus etc.

By equating the coefficients of t power r divided by r factorial we get r^{th} cumulant.
Thus,
K1 is equal to coefficient of t is equal to μ is equal to mu 1 dash, the mean
K2 is equal to coefficient of t square divided by 2 factorial is equal to 2 into lambda square, the variance.
K3 is equal to coefficient of t cube divided by 3 factorial is equal to zero, is equal to mu 3, and
K4 is equal to coefficient of t power 4 divided by 4 factorial is equal to 12 into lambda power 4.
Hence, mu 4 is equal to K4 plus 3 into K2 square is equal to 12 into lambda power 4 plus 3 into 2 lambda square whole square is equal to 24 into lambda power 4.
Hence, we can find coefficient of skewness and kurtosis.
Beta 1 is equal to mu 3 square divided by mu 2 cube is equal to zero and
Beta 2 is equal to mu 4 divided by mu 2 square is equal to 6.
Hence, Laplace distribution has symmetric and leptokurtic curve.

5. Uses and Applications of the Distribution

Laplace distribution has found a variety of specific uses, but they relate to the fact that it has long tails compared to the Normal Distribution. It has recently become quite popular in modeling financial variables (Brownian Laplace motion) like stock returns because of the greater tails. The Laplace distribution is very extensively reviewed in the monograph.

Following are some of the applications of Laplace distribution:

- The Laplacian distribution has been used in speech recognition to model priors on DFT coefficients. The addition of noise drawn from a Laplacian distribution, with scaling parameter appropriate to a function's sensitivity, to the output of a statistical database query is the most common means to provide differential privacy in statistical database

Here’s a summary of our learning in this session, where we understood:

- The Standard Laplace Distribution
    - Moments, skewness and kurtosis
    - mgf and cgf
- The Laplace distribution from exponential distribution
- The Laplace distribution with two parameters
    - Moments, mgf, and cgf
    - Skewness and kurtosis
- The uses and applications of the distribution