Frequently Asked Questions

1. Who discovered the Laplace distribution and when?

Answer:

The **Laplace distribution** is a continuous probability distribution named after Pierre-Simon Laplace. This distribution is often referred to as Laplace's first law of errors. He published it in 1774 when he noted that the frequency of an error could be expressed as an exponential function of its magnitude once its sign was disregarded.

2. Define standard Laplace distribution.

Answer:

A continuous random variable X is said to have Laplace (double exponential) distribution if its pdf is given by, $f(x) = \frac{1}{2}e^{-|x|}; -\infty < x < \infty$

3. Define Laplace distribution with two parameters.

Answer:

A continuous random variable X is said to have a double exponential (Laplace) distribution with two parameters λ and μ if it's pdf is given by, $f(x) = \frac{1}{2\lambda}e^{-\frac{|x-\mu|}{\lambda}}; -\infty < x < \infty, \lambda > 0$

4. Write the application of Laplace distribution.

Answer:

- The Laplacian distribution has been used in speech recognition to model priors on DFT coefficients.
- The addition of noise drawn from a Laplacian distribution, with scaling parameter appropriate to a function's sensitivity, to the output of a statistical database query is the most common means to provide differential privacy in statistical databases.

5. Mention the uses of Laplace distribution.

Answer:

It has recently become quite popular in modelling financial variables (Brownian Laplace motion) like stock returns because of the greater tails. The Laplace distribution is very extensively reviewed in the monograph.

6. Obtain an expression for r^{th} raw moment of standard Laplace distribution.

Answer:

If X is a standard Laplace variate, then rth raw moment is given by,

$\mu_r' = E(X^r)$

$$= \int_{-\infty}^{\infty} x^r f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} x^r e^{-|x|}dx = \frac{1}{2}\left[\int_{-\infty}^{0} x^r e^{-|x|}dx + \int_{0}^{\infty} x^r e^{-|x|}dx\right]$$

$$= \frac{1}{2}\left[(-1)^r\int_{0}^{\infty} x^r e^{-x}dx + \int_{0}^{\infty} x^r e^{-x}dx\right] = \frac{1}{2}\left[(-1)^r.\Gamma(r+1) + \Gamma(r+1)\right]$$

$$= \frac{1}{2}[\Gamma(r+1)\{(-1)^r + 1\}] = \frac{1}{2}r!\{(-1)^r + 1\}$$

7. Obtain first 4 moments from the general expression of moments.

Answer:

The rth raw moment is given by, $\frac{1}{2}r!\{(-1)^r + 1\}$.

By substituting r=1, we get,

Mean $\mu_1'= ½ .1!.[(-1)^1+1] = 0$
Since $\mu_1'=0$, all the moments about origin are same as the moments about the mean.
i.e., $\mu_r'= \mu_r$
Hence when r=2, $\mu_2'= \mu_2= ½ .2!.[(-1)^2+1] =2$
When r=3, $\mu_3'= \mu_3= ½ .3!.[(-1)^3+1] =0$ and
When r=4, $\mu_4'= \mu_4= ½ .4!.[(-1)^4+1] =24$.

8. Derive an expression for mgf of standard Laplace distribution.

Answer:

If a random variable X is a standard Laplace variate, then mgf is given by
$M_X(t)=E(e^{tX})$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{tx}e^{-|x|}dx = \frac{1}{2}\left[\int_{-\infty}^{0} e^{tx}e^{-|x|}dx + \int_{0}^{\infty} e^{tx}e^{-|x|}dx\right]$$

$$= \frac{1}{2}\left[\int_{0}^{\infty} e^{-tx}e^{-x}dx + \int_{0}^{\infty} e^{tx}e^{-x}dx\right] = \frac{1}{2}\left[\int_{0}^{\infty} e^{-x(1+t)}dx + \int_{0}^{\infty} e^{-x(1-t)}dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{(1+t)} + \frac{1}{(1-t)}\right] = \frac{1}{(1-t^2)} = (1-t^2)^{-1}$$

9. Find Cumulant generating function of standard Laplace distribution.

Answer:

Cumulant generating function of standard Laplace distribution is given by,

$K_X(t) = \log M_X(t)$

$= \log (1-t^2)^{-1}$

$= -\log(1-t^2) = -\left[(-t^2) - \frac{(-t^2)^2}{2} + \frac{(-t^2)^3}{3} - \ldots\right] = t^2 + \frac{t^4}{2} + \frac{t^6}{3} + \ldots$

10. Show that Standard Laplace distribution can be obtained as the distribution of difference of two independent exponential variates with parameter 1.

Answer:

If X and Y are independent with a common pdf $f(x)=e^{-x}$, $x\geq 0$, then the distribution of X-Y is standard Laplace variate.
We know that mgf of exponential variate with parameter θ is given by, $(1-t/\theta)^{-1}$ and when θ=1, mgf becomes $(1-t)^{-1}$.
Since X and Y are identically distributed,
$M_X(t)=M_Y(t)=(1-t)^{-1}$ and $M_{-Y}(t)=M_Y(-t)=(1+t)^{-1}$.
Hence $M_{X-Y}(t)=M_{X+(-Y)}(t)=M_X(t)\ M_{-Y}(t)=(1-t)^{-1}.\ (1+t)^{-1} =(1-t^2)^{-1}$, Which is the mgf of standard Laplace distribution. Hence by uniqueness theorem of mgf X-Y has Standard Laplace distribution.

11. Show that mean deviation about mean of a standard Laplace distribution is 1.

Answer:

Mean deviation about mean is given by,
MD = E(|X-0| = E(|X|)

$$= \int_{-\infty}^{\infty} |x| f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} |x| e^{-|x|}dx$$

$$= \frac{1}{2}.2\int_{0}^{\infty} xe^{-x}dx = \int_{0}^{\infty} x^{2-1}e^{-x}dx = \Gamma 2 = 1$$

12. Obtain an expression for finding the r^{th} raw moment of Laplace distribution.

Answer:

If X ~ Lap(λ,μ) the rth moment about origin is given by,
$\mu_r' = E(X^r)$

$$= \int_{-\infty}^{\infty} x^r f(x)dx = \frac{1}{2\lambda}\int_{-\infty}^{\infty} x^r e^{-\frac{|x-\mu|}{\lambda}} dx = \frac{1}{2}\int_{-\infty}^{\infty} (z\lambda + \mu)^r e^{-|z|}dz; z = \left(\frac{x-\mu}{\lambda}\right)$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\left[\sum_{k=0}^{r}\binom{r}{k}(z\lambda)^k \mu^{r-k}\right]e^{-|z|}dz = \frac{1}{2}\sum_{k=0}^{r}\left[\binom{r}{k}(\lambda)^k \mu^{r-k}\int_{-\infty}^{\infty} z^k e^{-|z|}dz\right]$$

$$= \frac{1}{2}\sum_{k=0}^{r}\left[\binom{r}{k}(\lambda)^k \mu^{r-k}\left\{\int_{-\infty}^{0} z^k e^{-|z|}dz + \int_{0}^{\infty} z^k e^{-|z|}dz\right\}\right]$$

$$= \frac{1}{2}\sum_{k=0}^{r}\left[\binom{r}{k}(\lambda)^k \mu^{r-k}\left\{(-1)^k\int_{0}^{\infty} z^k e^{-z}dz + \int_{0}^{\infty} z^k e^{-z}dz\right\}\right]$$

$$= \frac{1}{2}\sum_{k=0}^{r}\left[\binom{r}{k}(\lambda)^k \mu^{r-k}\Gamma(k+1)\left\{(-1)^k + 1\right\}\right] = \frac{1}{2}\sum_{k=0}^{r}\left[\binom{r}{k}(\lambda)^k \mu^{r-k}k!\left\{(-1)^k + 1\right\}\right]$$

13. Find mgf of Laplace distribution.

Answer:

Mgf of the Lap(λ,μ) is given by,
$M_X(t) = E(e^{tX})$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)dx = \frac{1}{2\lambda}\int_{-\infty}^{\infty} e^{tx} e^{-\frac{|x-\mu|}{\lambda}} dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{t(z\lambda+\mu)}e^{-|Z|}dz; \left(z = \frac{x-\mu}{\lambda}\right)$$

$$= \frac{e^{t\mu}}{2}\left[\int_{-\infty}^{0} e^{tz\lambda}e^{-|z|}dz + \int_{0}^{\infty} e^{tz\lambda}e^{-|z|}dz\right] = \frac{e^{t\mu}}{2}\left[\int_{0}^{\infty} e^{-tz\lambda}e^{-z}dz + \int_{0}^{\infty} e^{tz\lambda}e^{-z}dz\right]$$

$$= \frac{e^{t\mu}}{2}\left[\int_{0}^{\infty} e^{-z(1+t\lambda)}dz + \int_{0}^{\infty} e^{-z(1-t\lambda)}dz\right] = \frac{e^{t\mu}}{2}\left[\frac{1}{(1+t\lambda)} + \frac{1}{(1-t\lambda)}\right]$$

$$= \frac{e^{t\mu}}{(1-(t\lambda)^2)} = e^{t\mu}(1-t^2\lambda^2)^{-1}$$

14. Obtain cgf of Laplace distribution.

Answer:

Cumulant generating function
$K_X(t) = \log M_X(t)$

$= \log e^{t\mu}(1-t^2\lambda^2)^{-1}$

$= t\mu - \log(1-t^2\lambda^2)$

$$= t\mu - \left[(-t^2\lambda^2) - \frac{(-t^2\lambda^2)^2}{2} + \frac{(-t^2\lambda^2)^3}{3} - \ldots\right] = t\mu + t^2\lambda^2 + \frac{t^4\lambda^4}{2} + \frac{t^6\lambda^6}{3} + \ldots$$

15. Obtain first 4 moments of Laplace distribution from cgf and hence find coefficient of skewness and kurtosis.

Answer:

We know that cgf of the Laplace distribution is $K_X(t) = t\mu + t^2\lambda^2 + \frac{t^4\lambda^4}{2} + \frac{t^6\lambda^6}{3} + \ldots$

By equating the coefficients of $t^r/r!$ we get r^{th} cumulant.
Thus,
K_1=coefficient of t $=\mu=\mu_1'$=mean
K_2=coefficient of $t^2/2! = 2\lambda^2$ =Variance

K_3=coefficient of $t^3/3!=0=\mu_3$, and
K_4=coefficient of $t^4/4!=12\ \lambda^4$
Hence $\mu_4=K_4+3K_2^2=12\ \lambda^4+3(2\ \lambda^2)^2=24\ \lambda^4$
Hence, we can find coefficient of skewness and kurtosis.

$$\beta_1=\frac{\mu_3^2}{\mu_2^3}=0,\ \beta_2=\frac{\mu_4}{\mu_2^2}=6$$

Hence, Laplace distribution has symmetric and leptokurtic curve.