# Frequently Asked Questions

1. Write conditional mean of a continuous function g(X, Y) of discrete bivariate random variable.
   **Answer:**
   Conditional mean value or expectation of a continuous function g(X, Y) given that $Y=y_j$, is defined by

$$E[g(X,Y) \mid Y = y_j] = \sum_{i=1}^{\infty} g(x_i, y_j) P(X = x_i \mid Y = y_j)$$

$$= \frac{\sum_{i=1}^{\infty} g(x_i, y_j) P(X = x_i \cap Y = y_j)}{P(Y = y_j)}$$

2. Define Expectation of Discrete random variable.
   **Answer:**
   The conditional expectation of discrete random variable X given $Y=y_j$ is given by,

$$E[X \mid Y = y_j] = \sum_{i=1}^{\infty} x_i P(X = x_i \mid Y = y_j)$$

   The conditional expectation of discrete random variable Y given $X=x_i$ is given by,

$$E[Y \mid X = x_i] = \sum_{j=1}^{\infty} y_j P(Y = Y_j \mid X = x_i)$$

3. Give the conditional variance of discrete bivariate random variable.
   **Answer:**
   The conditional variance of X given $Y=y_j$ is given by,

$$V[X \mid Y = y_j] = E[\{X - E(X \mid Y = y_j]^2\} \mid Y = y_j]$$ and

   Conditional variance of Y given $X=x_i$ can be written as

$$V[Y \mid X = x_i] = E[\{Y - E(Y / X = x_i)^2\} \mid X = x_i]$$

4. Write conditional mean of a continuous function g(X, Y) of discrete bivariate random variable.
   **Answer:**
   Conditional mean value or expectation of a continuous function g(X, Y) given Y=y is given by,

$$E[g(X,Y) \mid Y = y] = \int_{-\infty}^{\infty} g(x,y) f_{X|Y}(x \mid y) dx = \frac{\int_{-\infty}^{\infty} g(x,y) f_{XY}(xy) dx}{f_Y(y)}$$

5. Write the conditional mean of continuous bivariate random variable.
   **Answer:**
   The conditional mean of X given Y=y is defined by,

$$E[X \mid Y = y)] = \frac{\int_{-\infty}^{\infty} x f_{XY}(xy) dx}{f_Y(y)} \quad \text{and} \quad E[Y \mid X = x)] = \frac{\int_{-\infty}^{\infty} y f_{XY}(x,y) dy}{f_X(x)}$$

6. Give conditional variance of continuous bivariate random variable.

**Answer:**
The conditional variance of X may be defined as,

$$V(X|Y=y)=E[\{X-E(X|Y=y)\}^2\ |Y=y]$$

Similarly, the conditional variance of Y is,

$$V(Y|X=x)=E[\{Y-E(Y|X=x)\}^2\ |X=x]$$

7. The expected value of X is equal to the expectation of the conditional expectation of X given Y. Symbolically, E(X)=E{E(X|Y)} for discrete bivariate random variables.

**Answer:**
Consider, $E\{E(X|Y)\} = E\{\sum_i x_i P(X=x_i|Y=y_j)\}$

$$= E\left\{\sum_i x_i \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}\right\} = \sum_j \left[\sum_i \left\{x_i \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}\right\}\right] P(Y = y_j)$$

$$= \sum_i \sum_j x_i P(X = x_i \cap Y = y_j) = \sum_i \left[x_i \left\{\sum_j P(X = xi \cap Y = y_j\right\}\right]$$

$$= \sum_i x_i P(X = x_i) = E(X)$$

8. The expected value of X is equal to the expectation of the conditional expectation of X given Y. Symbolically, E(X)=E{E(X|Y)} for discrete bivariate random variables.

**Answer:**
Consider

$$E\{E(X \mid Y)\} = E\left\{\int_{-\infty}^{\infty} x f(x \mid y) dx\right\} = E\left\{\int_{-\infty}^{\infty} x \frac{f(x,y)}{f_Y(y)} dx\right\}$$

$$= \int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} x \frac{f(x,y)}{f_Y(y)} dx\right\} f_Y(y) dy = \int_{-\infty}^{\infty} x \left\{\int_{-\infty}^{\infty} f(x,y) dy\right\} dx$$

$$= \int_{-\infty}^{\infty} x f_X(x) dx = E(X)$$

9. The variance of X can be regarded as consisting of two parts, the expectation of the conditional variance and the variance of the conditional expectation. Symbolically, V(X)=E[V(X|Y)]+V[E(X|Y)]

**Answer:**
Consider E[V(X|Y)]+V[E(X|Y)]

$$=E[E(X^2|Y)-\{E(X|Y)\}^2]+E[\{E(X|Y)\}^2]-[E\{E(X|Y)\}]^2$$

$$=E[E(X^2|Y)]-\cancel{E\{E(X|Y)\}^2+E\{E(X|Y)\}^2}-[E\{E(X|Y)\}]^2$$

$$=E[E(X^2|Y)]-\{E(X)\}^2$$

$$= E\left\{\sum_i x_i^2 P(X = x_i \mid Y = y_j)\right\} - \{E(X)\}^2$$

$$= \sum_j \left[\left\{\sum_i x_i^2 \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}\right\} P(Y = y_j)\right] - \{E(X)\}^2$$

$$= \sum_i \left\{x_i^2 \sum_j P(X = x_i \cap Y = y_j)\right\} - \{E(X)\}^2 = \sum_i x_i^2 P(X = x_i) - \{E(X)\}^2$$

$$=E(X^2)-\{E(X)\}^2=V(X).$$

Hence, the theorem.

10. Let A and B be two mutually exclusive events, then

$$E(X \mid AUB) = \frac{P(A)E(X \mid A) + P(B)E(X \mid B)}{P(AUB)},$$ Where by definition

$$E(X \mid A) = \frac{1}{P(A)} \sum_{x_i \in A} x_i P(X = x_i)$$

**Answer:**

$$E(X \mid AUB) = \frac{1}{P(AUB)} \sum_{x_i \in AUB} x_i P(X = x_i)$$

Since A and B are mutually exclusive events,

$$\sum_{x_i \in AUB} x_i P(X = x_i) = \sum_{x_i \in A} x_i P(X = x_i) + \sum_{x_i \in B} x_i P(X = x_i)$$

$$\therefore E(X \mid AUB) = \frac{1}{P(AUB)} \left[P(A)E(X \mid A) + P(B)E(X \mid B)\right]$$

11. Show that $E(X) = P(A)E(X \mid A) + P(\overline{A})E(X \mid \overline{A})$

**Answer:**

We know that,

$$\therefore E(X \mid AUB) = \frac{1}{P(AUB)} \left[P(A)E(X \mid A) + P(B)E(X \mid B)\right]$$

If we put $B = \overline{A}$ in the above theorem, we get,

$$E(X \mid AU\overline{A}) = \frac{1}{P(AU\overline{A})} \left[P(A)E(X \mid A) + P(\overline{A})E(X \mid \overline{A})\right]$$

$$\Rightarrow E(X) = P(A)E(X \mid A) + P(\overline{A})E(X \mid \overline{A})$$

12. Given two variables X and Y with joint density function f(x,y), prove that conditional mean of Y given X coincide with unconditional mean only if X and Y are independent.

**Answer:**

Conditional mean of Y give X is given by,

$E(Y \mid X) = \int_{-\infty}^{\infty} y f(y \mid x) dy$, where f(y|x) is conditional pdf of Y given X=x, which is given by, f(y|x)=f(x,y)/f(x)

Hence, $$E(Y \mid X) = \int_{-\infty}^{\infty} y \left[\frac{f(x,y)}{f(x)}\right] dy ----------(1)$$

Unconditional mean of Y is given by,

$$E(Y) = \int_{-\infty}^{\infty} y f(y) dy -------------(2)$$

From (1) and (2) we conclude that the conditional mean of Y given X will coincide with unconditional mean of Y only if

$$\frac{f(x,y)}{f(x)} = f(y) \Rightarrow f(x,y) = f(x)f(y).$$

**i.e.,** X and Y are independent.

13. Let f(x, y) =8xy, 0<x<y<1; and zero elsewhere. Find E(Y|X=x), E(XY|X=x), V(Y|X=x).

**Answer:**

First let us find the marginal pdf of given distribution of X

$$f_X(x) = \int_x^1 f(x,y)dy = 8x\int_x^1 y\,dy = 4x(1-x), 0 < x < 1$$

Conditional pdf of Y given X is given by,

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{2y}{1-x^2}; 0 < x < y < 1$$

$$E(Y \mid X = x) = \int_x^1 y\left[\frac{2y}{1-x^2}\right]dy = \frac{2}{3}\left[\frac{1-x^3}{1-x^2}\right] = \frac{2}{3}\left[\frac{1+x+x^2}{1+x}\right]$$

$$E(XY \mid X = x) = xE(Y \mid X = x) = \frac{2}{3}\frac{x(1+x+x^2)}{1+x}$$

To find V(Y|X=x), first let us find the $E(Y^2|X=x)$.

$$E(Y^2 \mid X = x) = \int_x^1 y^2\left[\frac{2y}{1-x^2}\right]dy = \frac{1}{2}\left[\frac{1-x^4}{1-x^2}\right] = \frac{1+x^2}{2}$$

$$\therefore V(Y \mid X = x) = E(Y^2 \mid X = x) - [E(Y \mid X = x)]^2$$

$$= \frac{1+x^2}{2} - \frac{4}{9}\frac{(1+x+x^2)^2}{(1+x)^2}$$

14. Let $f(x, y)=21x^2y^3$, 0<x<y<1 and zero elsewhere be the joint pdf of X and Y. Find the conditional mean and variance of X given Y=y, 0<y<1.

**Answer:**

Given $f(x, y)=21x^2y^3$, 0<x<y<1 and zero elsewhere. Marginal pdf of Y is given by,

$$f_Y(y) = \int_0^y f(x,y)dx = 21y^3\int_0^y x^2 dx = 7y^6$$

Therefore, the conditional pdf of X given Y is given by,

$$f(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{21x^2y^3}{7y^6} = 3\frac{x^2}{y^3}; 0 < x < y; 0 < y < 1$$

Conditional mean of X is,

$$E(X \mid Y = y) = \int_0^y x f(x \mid y)dx = \frac{3}{y^3}\int_0^y x^3 dx = \frac{3y}{4}; 0 < y < 1$$

To find variance, let us find,

$$E(X^2 \mid Y = y) = \int_0^y x^2 f(x \mid y)dx = \frac{3}{y^3}\int_0^y x^4 dx = \frac{3}{y^3}\frac{y^5}{5} = \frac{3}{5}y^2; 0 < y < 1$$

$$\therefore V(X|Y=y)=E(X^2|Y=y)-[E(X|Y=y)]^2 = \frac{3}{5}y^2 - \frac{9}{16}y^2 = \frac{3}{80}y^2; 0 < y < 1$$

15. Let X and Y be two random variable each taking three values -1, 0, 1 and having the following joint probability distribution, find V(Y|X=-1)

| Y \ X | **-1** | **0** | **1** | **Total** |
|---|---|---|---|---|
| **-1** | 0 | 0.1 | 0.1 | **0.2** |
| **0** | 0.2 | 0.2 | 0.2 | **0.6** |
| **1** | 0 | 0.1 | 0.1 | **0.2** |
| **Total** | **0.2** | **0.4** | **0.4** | **1.0** |

**Answer:**

We know that,

$V(Y|X=-1)=E(Y^2|X=-1) -[E(Y|X=-1)]^2$

$E(Y|X=-1)=\sum_y yP(Y=y,X=-1)=(-1)0+0(0.2)+1(0)=0$

$E(Y^2|X=-1)=\sum_y y^2P(Y=y,X=-1)=(-1^2)0+0^2(0.2)+1^2(0)=0$

$\therefore V(Y|X=-1)=0$