1. Introduction

Welcome to the series of E-learning module on normal distribution.

By the end of this session, you will be able to:

- Explain the normal distribution
- Explain the distribution function phi of Z
- Explain the limiting case of other distributions
- Explain mode and median of the distribution
- Explain Moment Generating Function (mgf), Cumulant Generating Function (cgf), Skewness and Kurtosis
- Explain the moments
- Explain the points of inflexion and mean deviation
- Explain the distribution of linear combination of normal variates

A random variable X is said to follow normal distribution with parameters mu and sigma square if its probability density function is given by,
f of x is equal to 1 divided by sigma into root 2 phi, e power minus half into x minus mu by sigma the whole square, x lies between minus infinity to infinity, mu lies between minus infinity to infinity and sigma is greater than zero.

Now, consider the following remarks:
1. When a random variable is normally distributed with mean mu and standard deviation sigma, it is customary to write X is distributed as Normal (mu, sigma square) and is expressed by X follows Normal (mu, sigma square).

2. If X follows Normal (mu, sigma square), then Z is equal to X minus mu divided by sigma is a standard normal variate with expectation of Z is equal to zero and Variance of Z is equal to 1 and we write Z follows normal zero, 1.
3. The probability density function of standard normal variate Z is given by,
f of z is equal to 1 divided by root 2 phi into e power minus z square divided by 2, z lies between minus infinity to infinity and corresponding distribution function is given by, phi of z is equal to probability that Z is less than or equal to z is equal to 1 divided by root 2 phi into integral from minus infinity to z e power minus z square by 2 dz.
4. The graph of f of (x) is a famous bell-shaped curve. The top of the bell is directly above the mean mu. For large values of sigma, the curve tends to flatten out and for small values of sigma, it has a sharp peak.

We shall prove two important results on the distribution function phi of standard normal variate.
The first one is,
Phi of minus z is equal to 1 minus phi of z, where z is greater than zero
To prove this, consider the left hand side,
Phi of minus z is equal to probability that Z is less than or equal to minus Z
Is equal to probability that Z is greater than or equal to z (because of symmetry about zero)
Is equal to 1 minus probability that z is less than or equal to z.

Is equal to 1 minus phi of z.

The second result is,
Probability that a less than or equal to X less than or equal to b is equal to phi of b minus mu by sigma minus phi of a minus mu by sigma, where X follows normal mu sigma square.
To prove this, consider probability that a less than or equal to X less than or equal to b
Is equal to probability that a minus mu by sigma is less than or equal to Z less than or equal to b minus mu by sigma, where Z is equal to X minus mu by sigma
Is equal to probability that Z is less than or equal to b minus mu by sigma minus probability that Z is less than or equal to a minus mu by sigma
Is equal to phi of b minus mu by sigma minus phi of a minus mu by sigma.

In the previous modules, we have already discussed the chief characteristics and properties of Normal Distribution. We have stated that most of the distributions tend to normal distribution.
For example, Binomial Distribution, which is discrete, tends to normal distribution under the following conditions:

- n, the number of trials is indefinitely large, that is as n tends to infinity
- Neither p nor q is very small

Poisson distribution tends to normal distribution, when the parameter lambda tends to infinity.
Gamma distribution tends to normal distribution for large values of alpha.

2. Mode and Median

Mode is the value of x for which f of (x) is maximum that is, mode is the solution of f dash of (x) is equal to zero and f double dash of (x) is less than zero.
For normal distribution with mean mu and standard deviation sigma,
Log f of x is equal to c minus 1 by 2 into sigma square into x minus mu whole square, where c is equal to log 1 by sigma into root 2 into phi, a constant.
Differentiating with respect to x we get,
1 divided by f of x into f dash of x is equal to 1 divided by sigma square into x minus mu

f dash of x is equal to 1 divided by sigma square into x minus mu into f of x
and f double dash of x is equal to minus 1 divided by sigma square into 1 into f of x plus x minus mu into f dash of x.
By substituting for f dash of x, we get,
Minus f of x divided by sigma square into 1 plus x minus mu whole square by sigma square.
By equating f dash of x is equal to zero, we get, x minus mu is equal to zero implies, x is equal to mu.
At the point x is equal to mu, we have
f double dash of x is equal to minus 1 divided by sigma square into f of x at x is equal to mu,
Is equal to minus 1 by sigma square into 1 by sigma into root 2 into phi, which is less than zero
Hence, x is equal to mu is the mode of the normal distribution.

Now, let us obtain median of the normal distribution.
If M is the median of the normal distribution, we have,
Integral from minus infinity to M f of x dx is equal to half
Implies, 1 divided by sigma into root 2 into phi, integral from minus infinity to M, e power minus x minus mu whole square divided by 2 into sigma square dx is equal to half
By splitting the range from minus infinity to M as minus infinity to mu and from mu to M, we get
1 divided by sigma into root 2 into phi, integral from minus infinity to mu, e power minus x minus mu whole square divided by 2 into sigma square dx plus 1 divided by sigma into root 2 into phi, integral from mu to M, e power minus x minus mu whole square divided by 2 into sigma square dx is equal to half.
But 1 divided by sigma into root 2 into phi, integral from minus infinity to Mu, e power minus x minus mu whole square divided by 2 into sigma square dx
By substituting z is equal to x minus mu divided by sigma, we get,
1 divided by root 2 into phi into integral from minus infinity to zero e power minus z square by 2 dz is equal to half.

Hence, by substitution we get,
Half plus 1 divided by sigma into root 2 into phi, integral from mu to M, e power minus x minus mu whole square divided by 2 into sigma square dx is equal to half
Implies 1 divided by sigma into root 2 into phi, integral from mu to M, e power minus x minus mu whole square divided by 2 into sigma square dx is equal to zero
That is mu is equal to M
Hence, for normal distribution Mean is equal to Median is equal to Mode is equal to mu.

3. MGF, CGF, Skewness, Kurtosis and Moments

The moment generating function is given by,
MX of t is equal to integral from minus infinity to infinity e power tx into f of x dx
Is equal to 1 by sigma into root 2 into phi, integral from minus infinity to infinity, e power t into x into e power minus 1 by 2 into sigma square into x minus mu square dx
Is equal to 1 by root 2 into phi, into integral from minus infinity to infinity e power t into mu plus t into sigma into e power minus half into z square dz, where z is equal to x minus mu by sigma
Is equal to e power t into mu into 1 by root 2 into phi, into integral from minus infinity to infinity, e power minus half into z square minus 2 into t into sigma into z dz
Is equal to e power t into mu into 1 by root 2 phi, integral from minus infinity to infinity, e power minus half into z minus sigma t whole square minus sigma into t the whole square dz

Is equal to e power t into mu plus t square into sigma square divided by 2 into 1 by root 2 into phi, into integral from minus infinity to infinity e power minus half into z minus sigma into t whole square dz
Is equal to e power t into mu plus t square into sigma square divided by 2 into 1 by root 2 into phi, into integral from minus infinity to infinity e power minus half into u square du, where u is equal z minus sigma t.
Is equal to e power t into mu plus t square into sigma square divided by 2.

Let us obtain Moment generating function of standard normal variate: If X follows Normal distribution with parameters (mu, and sigma square), then standard normal variate is, Z is equal to (X minus μ) divided by σ and its moment generating function is given by,
MZ of t is equal to e power minus t into mu by sigma into MX of (t by σ)
Is equal to e power minus t into mu by sigma into e power t by sigma into mu plus t by sigma whole square into sigma square divided by 2
Is equal to e power square by 2

Cumulant generating function is given by, KX of t is equal to log MX of t
Is equal to log e power t into mu plus t square sigma square by 2
Is equal to t into mu plus t square sigma square by 2
Thus, K1 is equal to mu, the mean
K2 is equal to sigma square, the variance
K3 is equal to zero and K4 is equal to zero and hence mu 4 is equal to k4 plus 3 into K2 square is equal to 3 sigma power 4.
Thus, beta 1 is equal to mu 3 square by mu 2 cube is equal to zero and Beta 2 is equal to mu 4 by mu 2 square is equal to 3.
Hence, normal distribution is symmetric and has a normal curve or mesokurtic curve.

Now, let us obtain the moments of the normal distribution.
It can be easily obtained by using moment generating function about mean.
We know that, MX of t is equal to e power t into mu plus t square into sigma square divided by

2
Implies, MX minus mu of t is equal to e power minus t into mu into MX of t
Is equal to e power minus t into mu into e power t into mu plus t square into sigma square divided by 2
Is equal to e power t square into sigma square divided by 2
Is equal to 1 plus t square into sigma square divided by 2 plus t square into sigma square divided by 2whole square divided by 2 factorial plus t square into sigma square divided by 2 whole cube divided by 3 factorial plus up to plus t square into sigma square divided by 2 whole power n divided by n factorial plus etc.
The coefficient of t power r divided by r factorial in the above expression gives mu r, the r^{th} moment about mean. Since there is no term with odd power of t in the above expression, all odd order moments about mean vanish.

That is, mu 2n plus 1 is equal to zero for n is equal to 1, 2, etc.
and mu 2n is equal to coefficient of t power 2n divided by 2n factorial.
Is equal to sigma power 2n into 2n factorial divided by 2 power n into n factorial
By expanding 2n factorial we get,
Sigma power 2n divided by 2 power n into n factorial into 2n into 2n minus 1 into 2n minus 2 into 2n minus 3 into up to into 5 into 4 into 3 into 2 into 1.
By separating odd and even terms in the above we get
Sigma power 2n divided by 2 power n into n factorial into 1 into 3 into 5 into up to 2n minus 1 into 2 into 4 into 6 into 2n minus 2 into 2n
By taking common 2 in the last term we get,
Sigma power 2n divided by 2 power n into n factorial into 1 into 3 into 5 into up to 2n minus 1 into 2 power n into 1 into 2 into 3 into up to n
On simplification we get,
1 into 3 into 5 into 2n minus 1 into sigma power 2n

4. Points of Inflexion and Mean Deviation

At the point of inflexion of the normal curve, we should have f double dash of x is equal to zero and f triple dash of x is not equal to zero.
From the solution of mode, we have,
f double dash of x is equal to minus f of x divided by sigma square into 1 plus x minus mu the whole square divided by sigma square.
f double dash of x is equal to zero implies,
1 plus x minus mu the whole square divided by sigma square is equal to zero
Implies, x is equal to mu plus or minus sigma.
It can be easily verified that the points x is equal to mu plus or minus sigma, f triple dash of x is not equal to zero. Hence, the points of inflexion of the normal curve are given by mu plus or minus sigma

For a normal distribution, we know that mean is equal to median is equal to mode is equal to mu. Hence, mean deviation at mu is given by,
M.D is equal to integral from minus infinity to infinity mod x minus mu into f of x dx
Is equal to 1 divided by sigma into root 2 phi, integral from minus infinity to infinity mod x minus mu into e power minus of x minus mu the whole square divided by 2 into sigma square dx
Is equal to sigma by root 2 phi, integral from mod z into e power minus z square by 2 dz, where z is equal to x minus mu divided by sigma
Since mod z and e power minus z square by 2 are even function, we can write integral from minus infinity to infinity as 2 times integral from zero to infinity. Hence, we write,
2 into sigma divided by root 2 phi into integral from zero to infinity mod z into e power minus z square by 2 dz

Since in the interval zero to infinity, mod z is equal to z, we have
M.D is equal to sigma into root 2 by phi into integral from zero to infinity, z into e power minus z square by 2 dz
Is equal to sigma into root 2 by phi into integral from zero to infinity, e power minus t dt where t is equal to z square by 2
Is equal to sigma into root 2 by phi into e power minus t divided by minus 1, ranges from zero to infinity
Is equal to sigma into root 2 by phi
Or approximately equal to 4 by 5 into sigma.

5. Distribution of Linear Combination

Let X_i, I is equal to 1, 2, up to n be n independent normal variates with mean mu i and variance sigma i square respectively. Then,
MXI of t is equal to e power t into mu i plus t square sigma i square divided by 2
The moment generating function of their linear combination summation a i into X i, where a_1, a_2, up to a n are constants, is given by
M summation over i a i X i of t is equal to product from i is equal to 1 to n M a i X i of t, since X_i's are independent.
Is equal to M X1 of a1 t into M X2 of a2 t into up to into M X n of a n t, since M c X of t is equal to M X of c t

Using mgf of normal distribution, let us obtain mgf of M a i X of t is equal to MX of a i t
Is equal to e power mu i a i t plus t square a i square sigma i square divided by 2
Therefore,
M Summation over i a i X i of t is equal to (e power mu1 a1 t plus t square into a1 square into sigma1 square divided by 2) into (e power mu2 a2 t plus t square into a2 square into sigma2 square divided by 2) up to into (e power mu n a n t plus t square into a n square into sigma n square divided by 2)
Is equal to e power summation over i from 1 to n, a i mu i into t plus t square into summation over i from 1 to n a i square into sigma i square divided by 2, which the mgf of a normal variate with mean summation a i mu i and variance summation a i square into sigma I square.
Hence, by uniqueness theorem of mgf, summation a i into X i follows normal distribution with parameters summation a i into mu i and summation a i square into sigma i square.

Here's a summary of our learning in this session, where we have understood:

- The normal distribution
- The distribution function phi of Z
- The limiting case of other distributions
- The mode and median of the distribution
- The mgf, cgf, skewness and kurtosis
- The moments
- The points of inflexion and mean deviation
- The distribution of linear combination of normal variates