Frequently Asked Questions

1. Define normal variate.

Answer:

A random variable X is said to follow normal distribution with parameters μ and σ^2 if its pdf is given by, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$

2. Define standard normal variate.

Answer:

If X~ N(μ, σ^2), then Z=(X-μ)/σ is a standard normal variate with E(Z)=0 and V(Z)=1 and we write Z~N(0,1)

3. Show that Φ(-z)=1-Φ(z), z>0.

Answer:

Φ(-z)=P(Z≤-z)=P(Z≥z)=1-P(Z≤z)=1-Φ(z).

4. Prove that $P(a \le X \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$ where X~N(μ,σ^2)

Answer:

$$P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le z \le \frac{b-\mu}{\sigma}\right), \left(Z = \frac{X-\mu}{\sigma}\right)$$

$$= P\left(Z \le \frac{b-\mu}{\sigma}\right) - P\left(Z \le \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

5. State the conditions under which binomial distribution tends to normal distribution.

Answer:

Binomial distribution, which is discrete, tends to normal distribution under the following conditions:

- n, the number of trials is indefinitely large, i.e. n→∞
- Neither p nor q is very small

6. Obtain mode of the distribution.

Answer:

Mode is the value of x for which f(x) is maximum, i.e. mode is the solution of f'(x)=0 and f''(x)<0

For normal distribution with mean μ and standard deviation σ,

$$\log f(x) = c - \frac{1}{2\sigma^2}(x-\mu)^2, c = \log\frac{1}{\sigma\sqrt{2\pi}}, \text{cons tant}$$

Differentiating w. r. t. x we get, $\frac{1}{f(x)} f'(x) = \frac{1}{\sigma^2}(x-\mu)$

$$f'(x) = \frac{1}{\sigma^2}(x-\mu)f(x) \text{ and } f''(x) = -\frac{1}{\sigma^2}[1.f(x) + (x-\mu)f'(x)]$$

$$= -\frac{f(x)}{\sigma^2}[1 + \frac{(x-\mu)^2}{\sigma^2}]$$

f'(x)=0⇒x-μ=0⇒x= μ.

At the point x= μ, we have $f''(x) = -\frac{1}{\sigma^2}[f(x)]_{x=\mu} = -\frac{1}{\sigma^2}\frac{1}{\sigma\sqrt{2\pi}} < 0$

Hence x= μ is the mode of the normal distribution.

7. Find median of the distribution.

Answer:

If M is the median of the normal distribution, we have,

$$\int_{-\infty}^{M} f(x)dx = \frac{1}{2} \Rightarrow \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{M} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \frac{1}{2}$$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\mu} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx + \frac{1}{\sigma\sqrt{2\pi}}\int_{\mu}^{M} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \frac{1}{2}$$

But $\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\mu} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-\frac{z^2}{2}}dz = \frac{1}{2}$

Hence, by substitution, $\frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}}\int_{\mu}^{M} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \frac{1}{2} \Rightarrow \frac{1}{\sigma\sqrt{2\pi}}\int_{\mu}^{M} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = 0$

i.e. μ=M

Hence for normal distribution Mean=Median=Mode=μ.

8. Derive mgf of normal distribution.

Answer:

The mgf is given by,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t(\mu+t\sigma)} e^{-\frac{1}{2}(z)^2}dz, \left(z = \frac{x-\mu}{\sigma}\right)$$

$$= e^{t\mu}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2-2t\sigma z)}dz = e^{t\mu}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left\{(z-\sigma t)^2-(\sigma t)^2\right\}}dz$$

$$= e^{t\mu+t^2\sigma^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2}dz = e^{t\mu+t^2\sigma^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(u)^2}du, (u = z-\sigma t)$$

$$= e^{t\mu+t^2\sigma^2/2}$$

9. Write the mgf of standard normal from the mgf of normal distribution.

Answer:

If X~N(μ, σ^2), then $M_X(t) = e^{t\mu+t^2\sigma^2/2}$

Standard normal variate is, Z=(X- μ)/σ and its mgf is given by,

$M_Z(t)=e^{-t\mu/\sigma} M_X(t/\sigma) = e^{-t\mu/\sigma}e^{(t/\sigma)\mu+\left\{(t/\sigma)^2\sigma^2\right\}/2} = e^{t^2/2}$

10. Find Cumulant generating function of normal variate.

Answer:

Cgf is given by, $K_X(t)=\log M_X(t) = \log(e^{t\mu+t^2\sigma^2/2}) = t\mu + t^2\sigma^2/2$

11. Write the nature of normal distribution.

Answer:

For normal distribution, $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0$, $\beta_2 = \frac{\mu_4}{\mu_2^2} = 3$

Hence, normal distribution is symmetric and has normal curve or mesokurtic curve.

12. Obtain the expression for central moments of normal distribution.

Answer:

It can be easily obtained by using moment generating function about mean.

We know that

$$M_X(t) = e^{t\mu + t^2\sigma^2/2}$$

$$\Rightarrow M_{X-\mu}(t) = e^{-t\mu}M_X(t) = e^{-t\mu}e^{t\mu + t^2\sigma^2/2} = e^{t^2\sigma^2/2}$$

$$= 1 + (t^2\sigma^2/2) + \frac{(t^2\sigma^2/2)^2}{2!} + \frac{(t^2\sigma^2/2)^3}{3!} + \ldots + \frac{(t^2\sigma^2/2)^n}{n!} + \ldots$$

The coefficient of $t^r/r!$ in the above expression gives μ_r, the r^{th} moment about mean. Since there is no term with odd power of t in the above expression, all odd order moments about mean vanish.

That is, $\mu_{2n+1}=0$, n=1, 2, …

And μ_{2n}=coefficient of $t^{2n}/2n!$

$$= \frac{\sigma^{2n} \times (2n)!}{2^n n!} = \frac{\sigma^{2n}}{2^n n!}[2n(2n-1)(2n-2)(2n-3)\ldots 5.4.3.2.1]$$

$$= \frac{\sigma^{2n}}{2^n n!}[1.3.5\ldots.(2n-1)][2.4.6\ldots(2n-2)2n]$$

$$= \frac{\sigma^{2n}}{2^n n!}[1.3.5\ldots.(2n-1)]2^n[1.2.3\ldots n]$$

$$= [1.3.5\ldots.(2n-1)]\sigma^{2n}$$

13. Find points of inflexion of normal distribution.

Answer:

At the point of inflexion the normal curve, we should have f"(x)=0 and f'"(x)≠0

For normal distribution with mean μ and standard deviation σ,

$$\log f(x) = c - \frac{1}{2\sigma^2}(x-\mu)^2, c = \log\frac{1}{\sigma\sqrt{2\pi}}, constant$$

Differentiating w. r. t. x we get, $\frac{1}{f(x)}f'(x) = \frac{1}{\sigma^2}(x-\mu)$

$f'(x) = \frac{1}{\sigma^2}(x-\mu)f(x)$ and $f''(x) = -\frac{1}{\sigma^2}[1.f(x) + (x-\mu)f'(x)]$

$$= -\frac{f(x)}{\sigma^2}[1 + \frac{(x-\mu)^2}{\sigma^2}]$$

f"(x)=0, $\Rightarrow [1 + \frac{(x-\mu)^2}{\sigma^2}] = 0 \Rightarrow x = \mu \pm \sigma$

It can be easily verified that the points x=μ±σ, f'"(x) ≠0. Hence the points of inflexion of the normal curve are given by, μ±σ

14. Obtain mean deviation about the mean.

Answer:

For a normal distribution, we know that mean=median=mode=μ. Hence mean deviation at μ is given by,

$$M.D = \int_{-\infty}^{\infty} | x - \mu | f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} | x - \mu | e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} | z | e^{-z^2/2} dz, \left(z = \frac{x-\mu}{\sigma} \right) = \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} | z | e^{-z^2/2} dz$$

Since in [0, ∞], |z|=z, we have

$$M.D = \sigma\sqrt{\frac{2}{\pi}} \int_0^{\infty} z e^{-z^2/2} dz = \sigma\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t} dt, (t = z^2/2)$$

$$= \sigma\sqrt{\frac{2}{\pi}} \left| \frac{e^{-t}}{-1} \right|_0^{\infty} = \sigma\sqrt{\frac{2}{\pi}} = \frac{4}{5}\sigma(approx.)$$

15. Obtain the distribution of linear combination of normal variates.

Answer:

Let X_i, i=1, 2 … n be n independent normal variates with mean μ_i and variance σ_i^2 respectively. Then $M_{X_i}(t) = e^{t\mu_i + t^2\sigma_i^2/2}$

The mgf of their linear combination $\Sigma a_i X_i$, where $a_1, a_2, \ldots, a_n$ are constants, is given by

$$M_{\sum_i a_i X_i}(t) = \prod_{i=1}^{n} M_{a_i X_i}(t) = M_{X_1}(a_1 t).M_{X_2}(a_2 t)\ldots M_{X_n}(a_n t)$$

Using the mgf of normal distribution, let us obtain mgf of

$$M_{a_i X}(t) = M_X(a_i t) = e^{\mu_i a_i t + t^2 a_i^2 \sigma_i^2/2}$$

Therefore

$$M_{\sum_i a_i X_i}(t) = e^{\mu_1 a_1 t + t^2 a_1^2 \sigma_1^2/2} e^{\mu_2 a_2 t + t^2 a_2^2 \sigma_2^2/2} \ldots e^{\mu_n a_n t + t^2 a_n^2 \sigma_n^2/2} = e^{\left[\sum_{i=1}^{n} a_i \mu_i\right] t + t^2 \left[\sum_{i=1}^{n} (a_i^2 \sigma_i^2)/2\right]},$$ which is the mgf of a normal variate with mean $\Sigma a_i \mu_i$ and variance $\Sigma a_i^2 \sigma_i^2$.

Hence, by uniqueness theorem of mgf, $\Sigma a_i X_i \sim N(\Sigma a_i \mu_i, \Sigma a_i^2 \sigma_i^2)$.