1. Introduction

Welcome to the series of E-learning modules on Marginal and Conditional Distributions.

By the end of this session, you will be able to:

- Explain the marginal probability mass function
- Explain the conditional probability mass function
- Explain the independence of two random variables
- Explain the marginal probability distribution function
- Explain the marginal density function
- Explain the conditional density function

Let (X, Y) be a discrete bivariate random variable which takes up countable number of values (x_i, y_i). Then, the probability distribution of X is determined as
pX of (x_i) is equal to Probability of (X is equal to x_i)
is equal to Probability of (X is equal to x_i intersection Y is equal to y_1) plus Probability of (X is equal to x_i intersection Y is equal to y_2) plus up to plus Probability of (X is equal to x_i intersection Y is equal to y_m)
is equal to p_{i1} plus p_{i2} plus up to plus p_{im}
is equal to summation over j is equal to 1 to m p ij is equal to summation over j is equal to 1 to m p xi, yj is equal to p i dot.
and is known as marginal probability mass function or discrete marginal density function of X.

Also, summation i is equal to 1 to n p i dot is equal to p1 dot plus p2 dot plus up to plus p n dot
Is equal to summation over i is equal to 1 to n, summation over j is equal to 1 to m p of xi, yj is equal to 1.
Similarly, we can prove that pY of yj is equal to probability of Y is equal to yj
Is equal to summation over i is equal to 1 to n, p ij is equal to summation over i is equal to 1 to n p of xi, yj is equal to p dot j, which is the marginal probability mass function of Y.

2. Conditional Probability Function

Now, let us define conditional probability function for discrete bivariate random variable.
Let (X, Y) be a discrete bivariate random variable. Then, the conditional discrete density function or the conditional probability mass function of X, given Y is equal to y denoted by p X given Y of (x given y), is defined as
p X given Y of x given y is equal to probability of X is equal to x, Y is equal to y, divided by probability of Y is equal to y, provided P (Y is equal to y) is not equal to zero.

Since for a fixed y, summation over i Probability of X is equal to xi, Y is equal to y divided by Probability of Y is equal to y
Is equal to 1 divided by probability of Y is equal to y into summation over i Probability of X is equal to xi, Y is equal to y
Is equal to 1 divided by Probability of Y is equal to y into Probability of Y is equal to y is equal to 1.
it follows that the conditional mass function p X given Y of (x given y) is a mass function, when considered as a function of the values of X.
The conditional probability mass function p Y given X of (y given x) is similarly defined.
That is, p Y given X of y given x is equal to probability of X is equal to x, Y is equal to y divided by probability X is equal to x, provided Probability of X is equal x is not equal to zero.

A necessary and sufficient condition for the discrete random variable X and Y to be independent is,
Probability of (X is equal to xi, Y is equal to yi) is equal to Probability of (X is equal to xi) into Probability of (Y is equal to yj) for all values of (xi, yj) of (X, Y)
That is the joint probability mass function can be expressed as the product of the conditional probability mass function.

3. Marginal Distribution & Density Function

So far we have found the marginal and conditional distribution of discrete random variables. Now, let us obtain the same for continuous random variable.
From the knowledge of joint distribution function F X Y of (x, y), it is possible to obtain the individual distribution functions F X of (x) and F Y of (y), which are termed as marginal distribution function of X and Y respectively with respect to the joint distribution function F X Y of (x, y).
F X of x is equal to probability of X less than or equal to x
Is equal to probability if X less than or equal to x, Y less than infinity
Is equal to limit as y tends to infinity F X Y of x, y
Is equal to F X Y of x and infinity.
Similarly,
F Y of y is equal to probability of Y less than or equal to y
Is equal to probability of X less than infinity, Y less than y
Is equal to limit as x tends to infinity F X Y of x, y
Is equal to F X Y of infinity and y.

Let (X, Y) be a continuous bivariate random variable with probability density function f X Y of(x, y). Then, marginal probability density function of X and Y are given respectively as
f X of x is equal to integral from minus infinity to infinity f X Y of x, y dy and
f Y of y is equal to integral from minus infinity to infinity f X Y of x, y dx.
Also, the marginal density functions of X and Y can be obtained in the following manner
f X of x is equal to d by dx of F X of x is equal to integral from minus infinity to infinity f X Y of x, y dy
f Y of y is equal to d by dy of F Y of y is equal to integral from minus infinity to f X Y of x, y dx

Consider the following important remark.
If we know the joint probability density function (or probability mass function) f X Y of (x, y) of two random variables X and Y, we can obtain the individual distribution of X and Y in the form of their marginal probability density functions (or probability mass functions) f X of (X) and f Y of (y). However, the converse is not true. That is from the marginal distributions of two jointly distributed random variables, we cannot determine the joint distribution of these two random variables.
Suppose, the variables are independent, then we can write joint probability density function (or probability mass function) as the product of their marginal probability density functions (or probability mass functions).

4. Conditional Distribution & Probability Density Function

Let us obtain conditional distribution function.
For a bivariate random variable (X, Y), the joint distribution function F X Y of (x, y) for any real numbers x and y is given by
F X Y of (x, y) is equal to Probability of (X less than or equal to x, Y less than or equal to y)
Now, let A be the event (Y less than or equal to y) such that event A is said to occur when Y assumes values up to and inclusive of y. Using conditional probabilities we may now write
F X Y of x, y is equal integral from minus infinity to x, Probability of A given X is equal to x d F X of x.

The conditional distribution function FY given X of (y given x) denotes the distribution function of Y when X has already assumed the particular value x. Hence,
F Y given X of (y given x) is equal to Probability of (Y less than or equal to y given X is equal to x) is equal to Probability of (A given X is equal to x)
Using this expression, the joint distribution function FXY(x, y) may be expressed in terms of the conditional distribution function as follows.
F X Y of x, y is equal to integral from minus infinity to x f Y given X of y given x d FX of x or we can write
F X Y of x, y is equal to integral from minus infinity to y, F X given Y of x given y d F Y of y

Now, let us discuss about conditional probability density function.
The conditional probability density function of Y given X for bivariate random variable (X, Y) which are jointly continuously distributed is defined as follows, for two real numbers x and y
f Y given X of (y given x) is equal to partial differentiation of F Y given X of y given x with respect to y.

Consider the following remarks.
If f X of (x) is greater than zero, then f Y given X of y given x is equal to f X Y of x, y divided by fX of x.
If fY(y) greater than 0, then f X given y is equal to f X Y of x, y divided by f Y of y
In terms of the differentials, we have
Probability of x less than X less than or equal to x plus dx given y less than Y less than or equal to y plus dy
Is equal to Probability of x less than X less than or equal to x plus dx, y less than Y less than or equal to y plus dy divided by probability of y less than y less than equal to y plus dy
Is equal to f X Y of x, y dx, dy divided by f Y of y dy
Is equal to f X given Y of x given y dx

Let us prove the following result.
Show that if f X of x is greater than zero, then f Y given X of y given x is equal to f X Y of x, y divided by f X of x
We have
F X Y of x, y is equal to integral from minus infinity to x f Y given X of y given x d F X of x

Is equal to integral from minus infinity to x f Y given X of y given x into f X of x dx
Differentiating with respect to x, we get
Dou by dou x of F X Y of x, y is equal to F Y given X of y given x into f X of x
Differentiating with respect to y, we get
Dou by dou y of Dou by dou x of F X Y of x, y is equal to f Y given X of y given x into f X of x
Implies f X Y of x, y is equal to f Y given X of y given x into f X of x
Therefore, f Y given X of y given x is equal to f X Y of x, y divided by f X of x.

5. Illustrations

Illustration - 1

Two discrete random variables X and Y have the joint probability mass function p X Y of (x, y) is equal to lambda power x into e power minus lambda into p power y into 1 minus p whole power x minus y divided by y factorial into x minus y factorial, where y take values zero, 1, 2 up to x and x take values zero, 1, 2 etc. where lambda and p are the constants with lambda greater than zero and zero less than p less than 1. Find

i. The marginal probability mass functions of X and Y
ii. The conditional distribution of Y for a given X and of X for a given Y

Solution:

The marginal probability mass function of X is given by,

p X of x is equal to summation over y is equal to zero to x p of x, y

Is equal to summation over y is equal to zero to x lambda power x into e power minus lambda into p power y into 1 minus p whole power x minus y divided by y factorial into x minus y factorial.

Taking the terms independent of y outside and then multiplying and dividing by x factorial we get,

Lambda power x into e power minus lambda divided by x factorial into summation over y is equal to zero to x, x factorial into p power y into 1 minus p power x minus y divided by y factorial into x minus y factorial.

Writing factorial terms as x c y, we get,

Lambda power x into e power minus lambda divided by x factorial into summation over y is equal to zero to x, x c x into p power y into 1 minus p power x minus y

Which is equal to Lambda power x into e power minus lambda divided by x factorial into p plus 1 minus p whole power x

Is equal to Lambda power x into e power minus lambda divided by x factorial, x take values zero 1, 2, etc., which is the probability function of a Poisson distribution with parameter lambda.

The marginal probability mass function of Y is given by

pY of y is equal to summation over x is equal to y to infinity p of x, y

is equal to summation over x is equal to y to infinity lambda power x into e power minus lambda into p power y into 1 minus p power x minus y divided by y factorial into x minus y factorial

Multiplying and dividing by lambda power y, we get

Lambda into p whole power y into e power minus lambda divided by y factorial into summation over x minus y is equal to zero to infinity lambda into 1 minus p whole power x minus y divided by x minus y factorial.

Is equal to lambda into p whole power y into e power minus lambda divided by y factorial into summation over t is equal to zero to infinity, lambda into 1 minus p whole power t divided by t factorial, where t is equal to x minus y

Is equal to lambda into p whole power y into e power minus lambda divided by y factorial into e power lambda into 1 minus p

Is equal to lambda into p whole power y into e power minus lambda into p divided by y factorial, which is the probability mass function of Poisson distribution with parameter lambda into p.

Let us find the conditional distribution of Y given X.
p of y given x is equal to p of x, y divided by p of x
is equal to lambda power x into e power minus lambda into p power y into 1 minus p power x minus y into x factorial divided by (y factorial into x minus y factorial into lambda power x into e power minus lambda)
is equal to p power y into 1 minus p power x minus y into x factorial divided by y factorial into x minus y factorial
is equal to x c y, p power y into 1 minus p power x minus y, where x is greater than or equal to y, that is y is equal to zero, 1, 2 up to x.

Let us find the conditional distribution of X given Y.
p x given y is equal to p of x, y divided by p of y
Is equal to lambda power x into e power minus lambda into p power y into 1 minus p power x minus y divided by y factorial into x minus y factorial, into y factorial divided by lambda into p whole power y into e power minus lambda into p
On simplification we get, e power lambda into q into lambda into q whole power x minus y divided by x minus y factorial, where q is equal to 1 minus p.
And range is given by x is greater than or equal to y, i.e. x is equal to y, y plus 1, y plus 2 etc.

Illustration -2
The joint probability density function of a two-dimensional random variable (X, Y) is given by, f of (x,y) is equal to 2; zero less than x less than 1, and zero less than y less than x
Find marginal distribution of X and Y and conditional distribution of X given Y is equal to y. Also verify whether X and Y are independent.

Solution
The Marginal distribution of X is given by,
f of x is equal to integral from minus infinity to infinity f of x, y dy
Is equal to integral from zero to x, 2 into dy
Is equal to 2 into x, where zero less than x less than 1

To find marginal distribution, range for Y is given as follows. We have zero less than x less than 1, and zero less than y less than x. Combining both we get, zero less than y less than x less than 1.
f of y is equal to integral from minus infinity to infinity, f of x, y dx
Is equal to integral from y to 1 2 into dx
Is equal to 2 into 1 minus y, where zero less than y less than 1.
Conditional distribution of X given Y is equal to y is given by,
f of x given by is equal to f of x, y divided by f of y is equal to 2 divided by 2 into 1 minus y
Is equal to 1 divided by 1 minus y, where y less than x less than 1.
X and Y are independent if f of (x, y) is equal to f(x) into f(y)
Consider f of x into f of y
Is equal to 2 x into 1 by 1 minus y, which is not equal to f of x, y.
Therefore, X and Y are not independent.

Here’s a summary of our learning in this session, where we have understood:

- The marginal Probability mass function
- The conditional probability mass function
- The independence of two random variables
- The marginal probability distribution function
- The marginal density function
- The conditional density function